

CLASSICAL MECHANICS

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HAMILTONIAN & LAGRANGIAN

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→ Degree of Freedom: minimum number of independent coordinates to configure the system.

1-Particle : 3-DOF

2-Particle : 5-DOF

3-Particle : 6-DOF

For Rigid Body : 6-DOF

For D-Space:

$$\text{DOF} = D + (D-1) + (D-2) + \dots + 1 + 0$$

$$\text{DOF} = \frac{D(D+1)}{2}$$

DOF in D-space for n-particles rigid system:

$$\text{DOF} = d + (d-1) + (d-2) + \dots + (d-(n-1))$$

$$= \frac{n}{2} [\text{first} + \text{last}] \quad n < d$$

$$\text{DOF} = \frac{n}{2} [2d - (n-1)]$$

→ Lagrangian Euler equation for conservative system:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

where $\frac{\partial L}{\partial \dot{q}} = p$ & $\frac{\partial L}{\partial q} = -\dot{p}$

- ③
- $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r, \theta, \phi) \rightarrow$ Spherical
- $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) - V(r, \phi, z) \rightarrow$ cylindrical
- $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r, \theta) \rightarrow$ Polar coordinates

Lagrangian equation in EM field:

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

generalised force:

$$Q = \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}} \right) - \frac{\partial U}{\partial q}$$

$$L = T - U$$

$U \rightarrow$ velocity dependent Potential.

$$L = \frac{1}{2} m v^2 + q(\vec{A} \cdot \vec{v}) - q\phi$$

$$L = \frac{1}{2} m v^2 - q(\phi - \vec{A} \cdot \vec{v})$$

Gauge Transformation:

$$\vec{A}' \rightarrow \vec{A} + \vec{\nabla} \lambda$$

$$\phi' \rightarrow \phi - \frac{\partial \lambda}{\partial t}$$

where $\lambda \rightarrow$ gauge function.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}$$

Ques 1 (4)
A particle of mass m moves inside a bowl.

If the surface of the bowl is given by the eqⁿ

$$z = \frac{1}{2} a (x^2 + y^2) \quad a \rightarrow \text{constant}$$

the Lagrangian of the particle is:

a) $\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 - g a r^2)$ b) $\frac{1}{2} m [(1 + a^2 r^2) \dot{r}^2 + r^2 \dot{\phi}^2]$

c) $\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - g a r^2)$ d) $\frac{1}{2} m [(1 + a^2 r^2) \dot{r}^2 + r^2 \dot{\phi}^2 - g a r^2]$

Solⁿo-

$$z = \frac{1}{2} a (x^2 + y^2) = \frac{1}{2} a r^2$$

$$\dot{z} = a r \dot{r}$$

$$x = r \cos \phi \Rightarrow \dot{x} = \dot{r} \cos \phi - r \sin \phi \dot{\phi}$$

$$y = r \sin \phi \Rightarrow \dot{y} = \dot{r} \sin \phi + r \cos \phi \dot{\phi}$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m g z$$

$$= \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\phi}^2 + a^2 r^2 \dot{r}^2] - m g \cdot \frac{1}{2} a r^2$$

$$= \frac{1}{2} m [(1 + a^2 r^2) \dot{r}^2 + r^2 \dot{\phi}^2 - g a r^2] \quad \square$$

Ques 2 The Lagrangian of particle of mass m moving in 1-Dim is given by:

$$L = \frac{1}{2} m \dot{x}^2 - b x \quad (b: \text{tve constant})$$

The co-ordinates of the particle $x(t)$ at time t is given by (c_1, c_2 are constants):

a) $-\frac{b}{2m}t^2 + C_1t + C_2$ b) $C_1t + C$

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c) $C_1 \cos\left(\frac{bt}{m}\right) + C_2 \sin\left(\frac{bt}{m}\right)$ d) none

Sol^{no}:- $L = T - V = \frac{1}{2}m\dot{x}^2 - bx$

eqn of motion:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x}) + b = 0$$

$$m\ddot{x} = -b$$

$$\frac{d^2x}{dt^2} = -\frac{b}{m}$$

$$\frac{dx}{dt} = -\frac{bt}{m} + C_1$$

$$x(t) = -\frac{b}{2m}t^2 + C_1t + C_2 \quad \boxed{A}$$

Que 3: The number of degree of freedom of a rigid body in d-space dimensions is:

a) $2d$ b) 6 c) $\frac{d(d+1)}{2}$ d) $d!$

Sol^{no}:- $\text{DOF} = d + (d-1) + (d-2) + \dots$

$$= \frac{d(d+1)}{2}$$

\boxed{C}

Ques 4 The equation of motion of a system described by the time-dependent Lagrangian: (6)

$$L = e^{rt} \left[\frac{1}{2} m \dot{x}^2 - V(x) \right] \text{ is}$$

a) $m\ddot{x} + r m \dot{x} + \frac{dV}{dx} = 0$

b) $m\ddot{x} + r m \dot{x} - \frac{dV}{dx} = 0$

c) $m\ddot{x} - r m \dot{x} + \frac{dV}{dx} = 0$

d) $m\ddot{x} + \frac{dV}{dx} = 0$

Solⁿ:- eqn of motion: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{a}} \right) - \frac{\partial L}{\partial q} = 0$

$$\frac{d}{dt} (e^{rt} m \dot{x}) - \frac{\partial}{\partial x} [e^{rt} V(x)] = 0$$

$$m \ddot{x} e^{rt} + r e^{rt} m \dot{x} + e^{rt} \frac{dV}{dx} = 0$$

$$m \ddot{x} + r m \dot{x} + \frac{dV}{dx} = 0 \quad \boxed{A}$$

Ques 5 The Lagrangian of a particle moving in a plane is given by

$$L = \dot{x}\dot{y} - x^2 - y^2$$

In polar co-ordinates the expression for the canonical momentum p_r is:

a) $\dot{r} \sin \theta + r \dot{\theta} \cos \theta$

b) $\dot{r} \cos \theta + r \dot{\theta} \sin \theta$

c) $2\dot{r} \cos \theta - r \dot{\theta} \sin \theta$

d) $\dot{r} \sin \theta + r \dot{\theta} \cos \theta$

Solⁿ:- $x = r \cos \theta$

$$\dot{x} = \dot{r} \cos \theta - r \sin \theta \dot{\theta}$$

$$y = r \sin \theta$$

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$$\dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta}$$

$$L = \dot{r}^2 \sin^2 \theta \cos^2 \theta + r \dot{r} \cos^2 \theta \dot{\theta} - r \dot{r} \sin^2 \theta \dot{\theta} - \cancel{r \dot{r} \cos^2 \theta \dot{\theta}} \\ - r^2 \dot{\theta}^2 \cos \theta \sin \theta - r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$\frac{\partial L}{\partial r} = p_r = 2 \dot{r} \sin^2 \theta \cos^2 \theta + r \cos^2 \theta \dot{\theta} - r \sin^2 \theta \dot{\theta}$$

$$= \dot{r} (2 \sin^2 \theta \cos^2 \theta) + r \dot{\theta} (\cos^2 \theta - \sin^2 \theta)$$

$$= \dot{r} \sin 2\theta + r \dot{\theta} \cos 2\theta \quad \square$$

Ques 6 The dynamics of a particle governed by the Lagrangian:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 - k x \dot{x} \quad \text{describes:}$$

- a) undamped H.O c) undamped H.O with
b) Damped H.O time dependent freq.
d) Free Particle.

solⁿ:- eqn of motion:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt} (m \dot{x} - k x \dot{x}) - (-k x - k \dot{x}^2) = 0$$

$$m \ddot{x} - k \dot{x}^2 - k x + k x + k \dot{x}^2 = 0$$

$$m \ddot{x} = 0$$

\square

POISSON BRACKET

⑧

Poisson Theorem: If u and v are constant of motion then their P.B with H also constant of motion.

$$[[u, v], H] = 0$$

Properties of Poisson Bracket:

1) $[A, B] = -[B, A] = AB - BA$

2) $[A, BC] = B[A, C] + [A, B]C$

3) $[A, B \pm C] = [A, B] \pm [A, C]$

4) $\frac{d}{dt}[A, B] = \left[\frac{dA}{dt}, B\right] + \left[A, \frac{dB}{dt}\right]$

5) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

6) $[L_i, L_j] = \epsilon_{ijk} L_k$

7) $[\gamma_i, L_j] = \epsilon_{ijk} \gamma_k$

8) $[p_i, L_j] = \epsilon_{ijk} p_k$

$$[\vec{F}, \vec{r} \cdot \hat{n}] = \hat{n} \times \vec{F}$$

↑
vector form
of 8)

ϵ_{ijk} $\begin{cases} \rightarrow +1 & \text{cyclic} \\ \rightarrow -1 & \text{anti-cyclic} \\ \rightarrow 0 & \text{other} \end{cases}$

Properties of Levi-Civita Tensor:

→ If two indices are equal: $\epsilon_{ijk} = 0$

e.g. $\epsilon_{121} = \epsilon_{311} = 0$

→ sign of two indices are interchangeable

i.e. $\epsilon_{ijk} = -\epsilon_{ikj}$

→ cyclic: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$

→ $\sum_{i,j,k}^3 \epsilon_{ijk} = 6$

→ Energy integral & Hamiltonian:

①

$$H = \sum \dot{q}_i p_i - L$$

$\frac{\partial L}{\partial \dot{q}} = p$	$\frac{\partial H}{\partial q} = -\dot{p}$
$\frac{\partial L}{\partial q} = \dot{p}$	$\frac{\partial H}{\partial p} = \dot{q}$

↑
Hamiltonian to Lagrangian
conversion and
vice-versa.

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Hamiltonian equation in P.B:

$$[X, Y]_{q,p} = \sum_i \left(\frac{\partial X}{\partial q_i} \frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial Y}{\partial q_i} \right)$$

Ques! write Hamiltonian equation in P.B?

solⁿ:- $[q, H] = \dot{q} = dq/dt$

$[p, H] = \dot{p} = dp/dt$

Ques! let A, B and C be the functions of phase space variables (co-ordinates & momenta of a mechanical system). If ξ, η represents the P.B, the value of $\{A, \{B, C\}\} - \{\{A, B\}, C\}$ is:

a) 0

b) $\{B, \{C, A\}\}$

c) $\{A, \{C, B\}\}$

d) $\{\{C, A\}, B\}$

solⁿ:- using Jacobis identity:

$$y = \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$$

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} = -\{B, \{C, A\}\} \quad (10)$$

$$\{A, \{B, C\}\} - \{\{A, B\}, C\} = \{\{C, A\}, B\} \quad \square$$

Ques 3: The coordinates and momenta x_i, p_i ($i=1, 2, 3$) of a particle satisfy the canonical P.B relation $\{x_i, p_j\} = \delta_{ij}$

$$\text{If } C_1 = x_2 p_3 + x_3 p_2 \quad \& \quad C_2 = x_1 p_2 - x_2 p_1$$

are constant of motion, & if $C_3 = \{C_1, C_2\} = x_1 p_3 + x_3 p_1$

then

$$a) \{C_2, C_3\} = C_1 \quad \& \quad \{C_3, C_1\} = C_2$$

$$b) \{C_2, C_3\} = -C_1 \quad \& \quad \{C_3, C_1\} = -C_2$$

$$c) \{C_2, C_3\} = -C_1 \quad \& \quad \{C_3, C_1\} = C_2$$

$$d) \{C_2, C_3\} = C_1 \quad \& \quad \{C_3, C_1\} = -C_2$$

$$\text{Sol}^{\text{no}}: \{C_2, C_3\} = \left(\frac{\partial C_2}{\partial x_1} \frac{\partial C_3}{\partial p_1} - \frac{\partial C_2}{\partial p_1} \frac{\partial C_3}{\partial x_1} \right) + \left(\frac{\partial C_2}{\partial x_2} \frac{\partial C_3}{\partial p_2} - \frac{\partial C_2}{\partial p_2} \frac{\partial C_3}{\partial x_2} \right) + \left(\frac{\partial C_2}{\partial x_3} \frac{\partial C_3}{\partial p_3} - \frac{\partial C_2}{\partial p_3} \frac{\partial C_3}{\partial x_3} \right)$$

$$= (p_2 x_3 - (-x_2) p_3)$$

$$= p_2 x_3 + x_2 p_3 = C_1$$

Similarly $\{C_3, C_1\} = -C_2$ □

Note:

$$\rightarrow \sum_{i,j,k=1}^3 \epsilon_{ijk} \{x_i, \{p_j, L_k\}\} = \sum \epsilon_{ijk}^2 = 6$$

$$\rightarrow \{r, p\} = \hat{\delta} \cdot \hat{p}$$

$$\rightarrow \sum \epsilon_{ijk} \cdot \epsilon_{ijl} = 2 \delta_{kl}$$

Ques 4 If the Lagrangian of a particle moving in one dimensional is given by: $L = \frac{1}{2} \dot{x}^2 - V(x)$ the Hamiltonian is: (11)

- a) $\frac{1}{2} x p^2 + V(x)$ b) $\frac{\dot{x}^2}{2} + V(x)$ c) $\frac{1}{2} \dot{x}^2 + V(x)$ d) none.

sol^{no} $H = \dot{x}p - L$

$$\frac{\partial L}{\partial \dot{x}} = p = \frac{\dot{x}}{1} \Rightarrow \dot{x} = xp$$

$$H = (xp)p - \frac{1}{2} (xp)^2 + V(x)$$

$$= xp^2 - \frac{xp^2}{2} + V(x)$$

$$= \frac{xp^2}{2} + V(x)$$

A

Ques 5 The Hamiltonian of a simple pendulum consisting of mass m attached to a massless string of length l is:

$$H = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$$

If L denotes the Lagrangian, the value of $\frac{dL}{dt}$ is:

a) $-\frac{2g}{l} p_\theta \sin\theta$

b) $-\frac{g}{l} p_\theta \sin\theta$

c) $\frac{g}{l} p_\theta \cos\theta$

d) $2 p_\theta^2 \cos\theta$

sol^{no} $L = \dot{\theta}p_\theta - H$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$L = \frac{p_\theta^2}{2me^2} + mgl(1 - \cos\theta)$$

$$p_\theta = 2me^2 \dot{\theta}$$

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$$L = 2me^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{dL}{dt} = [L, H] + \frac{\partial L}{\partial t} \rightarrow 0$$

$$[L, H] = \frac{\partial L}{\partial \theta} \frac{\partial H}{\partial p_\theta} - \frac{\partial L}{\partial p_\theta} \frac{\partial H}{\partial \theta}$$

$$= (-mgl \sin\theta) \frac{p_\theta}{2me^2} - \frac{p_\theta}{me^2} (mgl \sin\theta)$$

$$= -2mgl \sin\theta - \frac{p_\theta}{me^2}$$

$$= -2 \left(\frac{g}{l} \right) \sin\theta p_\theta \quad \boxed{A}$$

Que 6 The Hamiltonian of a relativistic particle of rest mass m and momentum p is given by:

$$H = \sqrt{p^2 + m^2} + V(x)$$

in units in which the speed of light $c=1$. The corresponding ~~Hamiltonian~~ Lagrangian is:

a) $L = m \sqrt{1 + \dot{x}^2} - V(x)$

b) $L = -m \sqrt{1 - \dot{x}^2} - V(x)$

c) $L = \sqrt{1 + m\dot{x}^2} - V(x)$

d) $L = \frac{1}{2} m \dot{x}^2 - V(x)$

Soln^o

$$H = \sqrt{p^2 + m^2} + V(x)$$

(13)

$$\frac{\partial H}{\partial p} = \dot{x} = \frac{1}{2} \frac{\partial p}{\sqrt{p^2 + m^2}} = \frac{p}{\sqrt{p^2 + m^2}} \quad \text{--- (1)}$$

$$L = \dot{x}p - H$$

$$L = \dot{x} \frac{p^2}{\sqrt{p^2 + m^2}} - \sqrt{p^2 + m^2} - V(x)$$

$$L = \frac{p^2 - (p^2 + m^2)}{\sqrt{p^2 + m^2}} - V(x) \Rightarrow \frac{-m^2}{\sqrt{p^2 + m^2}} - V(x) \quad \text{--- (2)}$$

by eqn (1)

$$p^2 + m^2 = \frac{p^2}{\dot{x}^2} \Rightarrow \dot{x}^2 p^2 - p^2 + m^2 \dot{x}^2 = 0$$

$$p^2(1 - \dot{x}^2) = m^2 \dot{x}^2$$

$$p^2 = \frac{m^2 \dot{x}^2}{1 - \dot{x}^2} \quad \text{--- (2)}$$

$$\frac{p^2}{m^2} = \frac{\dot{x}^2}{1 - \dot{x}^2} \quad \text{--- (3)}$$

eqn (2) becomes:

$$L = \frac{-m^2}{m \sqrt{\frac{p^2}{m^2} + 1}} - V(x) = \frac{-m}{\sqrt{\frac{\dot{x}^2}{1 - \dot{x}^2} + 1}} - V(x)$$

$$= \frac{-m}{\sqrt{\frac{\dot{x}^2 + 1 - \dot{x}^2}{1 - \dot{x}^2}}} - V(x)$$

$$L = -m\sqrt{1-\dot{x}^2} - V(x)$$

[B]

(14)

Que 7 A particle of mass m and co-ordinate q has the Lagrangian:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{\lambda}{2}q\dot{q}^2 \quad \lambda \rightarrow \text{constant.}$$

Then corresponding Hamiltonian is:

a) $\frac{p^2}{2m} + \frac{\lambda p^2 q}{2m^2}$ b) $\frac{p^2}{2(m-\lambda q)}$

c) $\frac{p^2}{2m} + \frac{\lambda q p^2}{2(m-\lambda q)^2}$ d) $\frac{p^2}{2}$

Solⁿo- $H = \dot{q}p - L$

$$\frac{\partial L}{\partial \dot{q}} = p = m\dot{q} - \lambda q \dot{q}$$

$$p = (m - \lambda q) \dot{q} \Rightarrow \dot{q} = \left(\frac{p}{m - \lambda q} \right) \text{--- (1)}$$

$$H = (m\dot{q} - \lambda q \dot{q}) \dot{q} - \frac{1}{2}m\dot{q}^2 - \frac{\lambda}{2}q\dot{q}^2$$

$$H = \frac{1}{2}m\dot{q}^2 - \frac{\lambda q \dot{q}^2}{2}$$

$$H = \left(\frac{m}{2} - \frac{\lambda q}{2} \right) \left(\frac{p}{m - \lambda q} \right)^2 \quad \text{by eqn (1)}$$

$$H = \left(\frac{m - \lambda q}{2} \right) \frac{p^2}{(m - \lambda q)^2} \Rightarrow \frac{p^2}{2(m - \lambda q)} \quad \text{[B]}$$

CONSERVATION OF MOTION

(15)

Energy conservation:

→ L is explicitly independent of time

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \quad H = \text{constant}$$

→ Potential does not contain velocity term

$$V = V(q) \quad \& \quad L = L(q, \dot{q})$$

Linear momentum conservation:

→ If a co-ordinate is cyclic then corresponding conjugate momentum is conserved.

$$\text{i.e. } \frac{\partial L}{\partial q} \text{ or } \frac{\partial H}{\partial q} = 0 \Rightarrow p_\phi = \text{constant.}$$

→ $\phi \rightarrow$ absent & $\dot{\phi} \rightarrow$ present then
 $p_\phi = \text{constant of motion}$

Angular momentum conservation:

→ $[L_x, V(x)] = 0 \quad L_x = \text{constant of motion}$

$[p_x, V(x)] = 0 \quad p_x = \text{constant of motion}$

example: L_x & L_y are constant of motion.

their P.B: $[L_x, L_y] \rightarrow$ also constant of motion.

$[L_x, L_y] = L_z \rightarrow$ also const. of motion.

Que! A system is governed by the Hamiltonian (16)

$$H = \frac{1}{2} (p_x - ay)^2 + \frac{1}{2} (p_y - bx)^2$$

For what values of a & b will the quantities $(p_x - 3y)$ & $(p_y + 2x)$ are conserved?

a) $a = -3, b = 2$

b) $a = 3, b = 2$

c) $a = 2, b = 3$

d) $a = -2, b = 3$

Solⁿo. $(p_x - 3y) = \alpha$
 $(p_y + 2x) = \beta$ } \rightarrow const. of motion.

By Poisson thm: $\{\alpha, H\} = 0$; $\{\beta, H\} = 0$

$$\{\alpha, H\} = \left(\frac{\partial \alpha}{\partial x} \cdot \frac{\partial H}{\partial p_x} - \frac{\partial \alpha}{\partial p_x} \frac{\partial H}{\partial x} \right) + \left(\frac{\partial \alpha}{\partial y} \frac{\partial H}{\partial p_y} - \frac{\partial \alpha}{\partial p_y} \frac{\partial H}{\partial y} \right) = 0$$

$$\Rightarrow 0 - 1 \cdot \frac{1}{2} \cdot 2 (p_y - bx) b + (-3) \cdot \frac{1}{2} (p_y - bx) \cdot 2 = 0$$

$$(p_y - bx) (b - 3) = 0$$

$$\Rightarrow b = 3$$

similarly

$$\{\beta, H\} = 0 \Rightarrow a = -2$$

(D)

Que! A particle moves in a potential

$$U = x^2 + y^2 + \frac{z^2}{2}$$

which component(s) of the angular momentum is/are constant(s) of motion?

a) none

b) L_x, L_y, L_z

(17)

c) L_x & L_y

d) L_z

Sol^{no} $x = r \cos\phi \sin\theta$, $y = r \cos\phi \cos\theta$, $z = r \cos\theta$

$$L = \frac{1}{2} m (\dot{x}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) - (r^2 + \frac{z^2}{2})$$

$\dot{\phi} \rightarrow$ Present

$\phi \rightarrow$ absent

p_ϕ or L_z is constant of motion.

example:

$$L = \frac{1}{2} m \dot{x}_1^2 + m (\dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2} k x_1^2 - \frac{1}{2} k (x_2 + x_3)^2$$

↑
Potential term.

→ $V(x) \rightarrow$ independent of velocity

⇒ $H =$ constant of motion.

→ none of the co-ordinate is cyclic.

$\frac{\partial L}{\partial q} \neq 0 \Rightarrow$ linear momentum is not conserved.

→ no $x_1^2 + x_2^2$ / $x_2^2 + x_3^2$ / $x_3^2 + x_1^2$ term

⇒ no component of angular momentum is conserved.

CANONICAL TRANSFORMATION

(18)

→ Poisson Bracket is invariant under canonical transformation.

$$[Q, P]_{q,p} = [q, p]_{Q,P}$$

→ condition for canonical transformation:

$$\boxed{[Q, P] = 1}$$

Que 1 A Hamiltonian system is described by the canonical coordinates q & canonical momentum p . A new co-ordinate Q is defined by

$$Q(t) = q(t+\tau) + p(t+\tau)$$

where t is the time & τ is a constant, i.e. the new coordinate is a combination of the old coordinate & momentum at a shifted time. The new ~~co-ordinate~~ momenta $P(t)$ can be expressed as:

a) $p(t+\tau) - q(t+\tau)$

b) $p(t+\tau) - q(t-\tau)$

c) $\frac{1}{2} [p(t-\tau) - q(t+\tau)]$

d) $\frac{1}{2} [p(t+\tau) - q(t+\tau)]$

Solⁿ For canonical transformation:

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} = 1$$

check option: let $P(t) = \frac{1}{2} [p(t+\tau) - q(t+\tau)]$

$$\{Q, P\}_{q, p} = 1 \cdot \underbrace{(1+0)}_{\substack{\rightarrow \text{neglected.} \\ \uparrow \\ f(x^n)}} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$$

(19)

$$= \frac{1}{2} + \frac{1}{2} = \textcircled{1} \text{ i.e. } \square \\ \text{canonical.}$$

Quesst Let q & p be the canonical co-ordinates & momentum of dynamical system. which of the following transformations is canonical.

$$A: Q_1 = \frac{1}{\sqrt{2}} q^2 \quad \& \quad P_1 = \frac{1}{\sqrt{2}} p^2$$

$$B: Q_2 = \frac{1}{\sqrt{2}} (p+q) \quad \& \quad P_2 = \frac{1}{\sqrt{2}} (p-q)$$

a) neither A nor B

b) both A & B

c) only A

d) only B

Solⁿ:- $[Q_1, P_1] = \left[\frac{1}{\sqrt{2}} q^2, \frac{1}{\sqrt{2}} p^2 \right]$

$$= \frac{1}{2} [q^2, p^2] \neq 0 \quad \text{not canonical}$$

i.e. $[q^2, p^2] = 2q [q, p^2] \neq 1.$

$$[Q_2, P_2] = \frac{1}{2} [p+q, p-q]$$

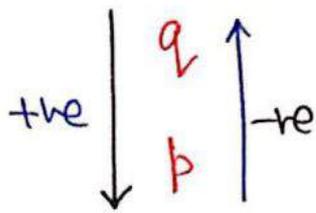
$$= \frac{1}{2} ([p, p] - [p, q] + [q, p] - [q, q])$$

$$= \frac{1}{2} [0 - (-1) + 1 + 0] = 1$$

\square

GENERATING FUNCTION (F)

Ⓢ



$$\frac{\partial F}{\partial q} = +p$$

$$\frac{\partial F}{\partial Q} = -P$$

$F =$ (half old,
half new)

$$\frac{\partial F}{\partial p} = -q$$

$$\frac{\partial F}{\partial P} = +Q$$

new Hamiltonian: $H' = H + \frac{\partial F}{\partial t}$

Que! A canonical transformation relates the old co-ordinates (q, p) to the new ones (P, Q) by the relations

$$Q = q^2 \text{ and } P = p/q$$

The corresponding time-independent generating function is:

a) p/q^2 b) $q^2 p$ c) q^2/p d) $q p^2$

Solⁿo - F is of form: $F(q, P)$

$$\text{Use } \frac{\partial F}{\partial P} = Q = q^2$$

$$\partial F = q^2 \partial P \Rightarrow F = q^2 P + C_1$$

$$P = \frac{p}{q} \Rightarrow p = q P$$

$$\frac{\partial F}{\partial q} = p = q P$$

$$\int \delta F = \int \delta q P \delta q$$

(21)

$$F = q^2 P + C_2$$

$$q = Q = 0$$

$$F = q^2 P$$

[B]

Ques A mechanical system is described by the Hamiltonian: $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$

As a result of the canonical transformation generated by $F(q, Q) = -Q/q$, the Hamiltonian in the co-ordinates Q & momentum P becomes:

a) $\frac{1}{2m} Q^2 P^2 + \frac{m \omega^2}{2} Q^2$

b) $\frac{1}{2m} Q^2 P^2 + \frac{m \omega^2}{2} P^2$

c) $\frac{1}{2m} P^2 + \frac{m \omega^2}{2} Q^2$

d) $\frac{1}{2m} Q^2 P^4 + \frac{m \omega^2}{2} P^2$

Soln^o $F(q, Q) = -Q/q$

$$\frac{\partial F}{\partial q} = p = \frac{Q}{q^2} \quad \& \quad \frac{\partial F}{\partial Q} = -P = -\frac{1}{q}$$

$$p = \frac{Q}{q^2} \quad \& \quad P = \frac{1}{q}$$

$$p = \frac{Q P^2}{P^2} \quad \text{now} \quad H' = \frac{\partial F}{\partial t} + H$$

$$H'(Q, P) = H(q, p) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$H'(Q, P) = \frac{(QP^{\alpha})^{\alpha}}{2m} + \frac{1}{2} m \omega^{\alpha} \left(\frac{1}{P}\right)^{\alpha}$$

22

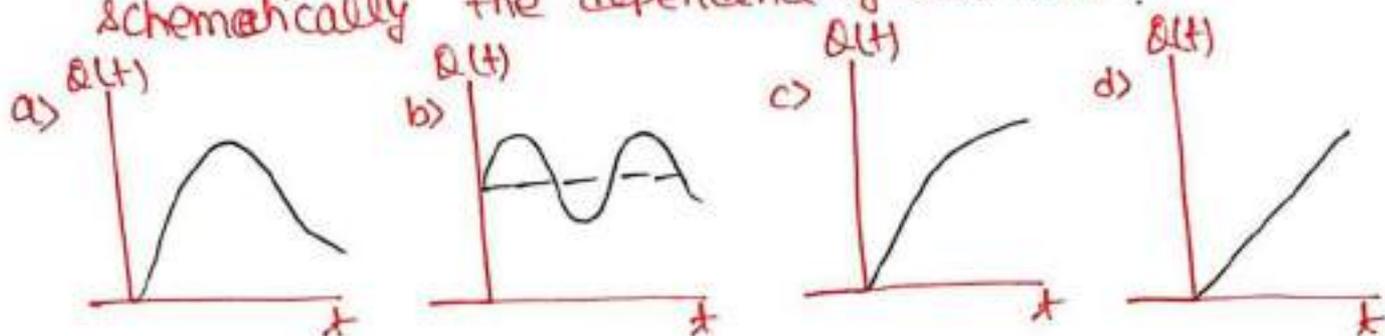
$$= \frac{1}{2m} Q^{\alpha} P^{\alpha} + \frac{1}{2} m \omega^{\alpha} \bar{P}^{\alpha} \quad \square$$

Que 3 A canonical transformation $(p, q) \rightarrow (P, Q)$ is performed on the Hamiltonian

$$H = \frac{p^{\alpha}}{2m} + \frac{1}{2} m \omega^{\alpha} q^{\alpha}$$

via the generating function $(F) = \frac{1}{2} m \omega q^{\alpha} \cot Q$

If $Q(0) = 0$, which of the following graphs shows schematically the dependence of $Q(t)$ on t ?



Solⁿ

$$F(q, Q) = \frac{1}{2} m \omega q^{\alpha} \cot Q$$

$$q = \frac{\partial F}{\partial q} = p = m \omega q \cot Q$$

$$\frac{\partial F}{\partial Q} = -P = -\frac{1}{2} m \omega q^{\alpha} \operatorname{cosec}^{\alpha} Q$$

$$\frac{1}{2} m \omega^{\alpha} q^{\alpha} = \frac{P \omega}{\operatorname{cosec}^{\alpha} Q}$$

$$H' = H + \frac{\partial F}{\partial t} = \frac{p^{\alpha}}{2m} + \frac{1}{2} m \omega^{\alpha} q^{\alpha} + 0$$

$$H' = \left(\frac{m^2 \omega^2 q^2}{2m} \right) \cot^2 \theta + \frac{P\omega}{\operatorname{cosec}^2 \theta}$$

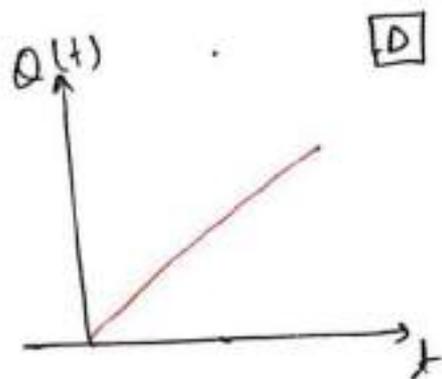
$$H' = \frac{P\omega}{\operatorname{cosec}^2 \theta} \cot^2 \theta + \frac{P\omega}{\operatorname{cosec}^2 \theta}$$

$$= P\omega \left[\frac{1 + \cot^2 \theta}{\operatorname{cosec}^2 \theta} \right] = \omega P$$

Hamiltonian eqn:

$$\dot{Q}(t) = \frac{\partial H'}{\partial P} = \omega$$

$$Q = \omega t + C$$



Que 4 A canonical transformation $(q, p) \rightarrow (Q, P)$ is made through the generating function:

$$F(q, p) = q^2 p$$

on the Hamiltonian:

$$H(q, p) = \frac{p^2}{2\alpha q^2} + \frac{\beta}{4} q^4$$

The eqn of motion for (Q, P) are:

$$a) \dot{Q} = \frac{P}{\alpha} \quad \& \quad \dot{P} = -\beta Q \qquad b) \dot{Q} = \frac{4P}{\alpha} \quad \& \quad \dot{P} = -\frac{\beta Q}{2}$$

$$c) \dot{Q} = \frac{P}{\alpha} \quad \& \quad \dot{P} = -\frac{2P^2}{\alpha} - \beta Q \qquad d) \dot{Q} = \frac{2P}{\alpha} \quad \& \quad \dot{P} = -\beta Q$$

Solⁿo-

$$F(q, p) = q^2 p$$

(24)

$$\frac{\partial F}{\partial q} = p = 2q p$$

$$H' = H + \frac{\partial F}{\partial t} \rightarrow 0$$

$$\frac{\partial F}{\partial p} = Q = q^2$$

$$H' = H$$

$$H' = \frac{p^2}{2 \times q^2} + \frac{\beta}{4} q^4$$

$$= \frac{4q^2 p^2}{2 \times q^2} + \frac{\beta}{4} Q^2 \Rightarrow \frac{2p^2}{\alpha} + \frac{\beta}{4} Q^2$$

now

$$\dot{Q} = \frac{\partial H'}{\partial p} = \frac{4p}{\alpha}$$

$$\dot{p} = -\frac{\partial H'}{\partial Q} = -\frac{\beta}{2} Q$$

\square

SMALL OSCILLATION

25

→ Frequency of small oscillation:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{V''(a)}{m}}$$

Types:

1) Find stable & unstable points:

$V(x)$: given

→ $V'(x) = 0$

→ Find x_1 & x_2

→ $V''(x) > 0$ +ve minima (stable)
 < 0 -ve maxima (unstable)

2) Find equilibrium distance

→ $V'(x) = 0$

→ Find x_0

→ Find $V(x) \Big|_{x=x_0}$

Que 1 A particle of unit mass moves in a potential

$V(x) = ax^2 + \frac{b}{x^2}$. The angular frequency of small

oscillations about the minimum of the potential is:

a) $\sqrt{8b}$ b) $\sqrt{8a}$ c) $\sqrt{8a/b}$ d) $\sqrt{8b/a}$

Solⁿo- $V'(x) = 2ax - \frac{2b}{x^3} = 0$

$$2ax_0 = \frac{2b}{x_0^3} \Rightarrow x_0^2 = \sqrt{\frac{b}{a}}$$

$$\begin{aligned} V''(x) \Big|_{x=x_0} &= 2a + \frac{6b}{x_0^4} \\ &= 2a + \frac{6b}{b/a} = 8a \end{aligned}$$

$$\omega = \sqrt{\frac{V''(x)}{m}} = \sqrt{\frac{8a}{1}} \Rightarrow \sqrt{8a} \quad \boxed{B}$$

Que 2 Consider the motion of a classical particle in a 1-Dim. double well potential $V(x) = \frac{1}{4}(x^2 - 2)^2$. If the particle is displaced infinitesimally from the minimum in the +ve x-axis (neglect friction) then,

- Particle will execute S.H.M in right well with an angular frequency $\omega = \sqrt{2}$
- Particle will execute S.H.M in right well with an angular frequency $\omega = 2$
- Particle will switch b/w right & left wells.
- none

Sol^{no} $V(x) = \frac{1}{4}(x^2 - 2)^2$

$$V'(x) = \frac{2}{4}(x^2 - 2)(2x) \Rightarrow (x^2 - 2)x = 0$$

$$x = 0, \pm\sqrt{2}$$

$$V''(x) = 3x^2 - 2$$

(27)

at $x=0$: $V''(x) = -2$ (unstable)

$x = \pm\sqrt{2}$: $V''(x) = 4$ (stable point).

$$\omega = \sqrt{\frac{4}{1}} = 2 \quad \boxed{B}$$

Que 3 A particle of mass m is at the stable equilibrium position of its potential energy

$$U(x) = ax - bx^3$$

The minimum velocity that has to be imparted to the particle to render its motion unstable is:

a) $\left(\frac{64a^3}{9m^2b}\right)^{1/4}$ b) $\left(\frac{64a^3}{27m^2b}\right)^{1/4}$ c) $\left(\frac{16a^3}{27m^2b}\right)^{1/4}$ d) none

Solⁿ:- $\frac{dV}{dx} \Big|_{x=x_0} = a - 3bx_0^2 = 0$

$$x = \pm\sqrt{\frac{a}{3b}}$$

$\frac{d^2V}{dx^2} = -6bx_0$ stable point ($x = -\sqrt{\frac{a}{3b}}$)

$$V(x_0) = ax_0 - bx_0^3 = x_0(a - bx_0^2)$$

$$= -\sqrt{\frac{a}{3b}} \left(a - b \cdot \frac{a}{3b}\right)$$

$$= -\sqrt{\frac{a}{3b}} \cdot \frac{2a}{3}$$

To make motion unstable

loss in K.E = gain in P.E

$$\frac{1}{2}mv^2 = V(x) \Big|_{x=\sqrt{\frac{a}{3b}}} - V(x) \Big|_{x=-\sqrt{\frac{a}{3b}}}$$

$$\frac{1}{2}mv^2 = \frac{2a}{3} \sqrt{\frac{a}{3b}} + \frac{2a}{3} \sqrt{\frac{a}{3b}}$$

$$v^2 = \frac{8a}{3m} \sqrt{\frac{a}{3b}}$$

$$v = \left(\frac{64a^3}{27m^2b} \right)^{1/4} \quad \boxed{B}$$

Ques The time evolution of 1-Dimensional dynamical system is described by:

$$\frac{dx}{dt} = -(x+1)(x^2-b^2)$$

If this has one stable & two unstable fixed points, then the parameter 'b' satisfy:

- a) $0 < b < 1$ b) $b > 1$ c) $b < -1$ d) $b = 2$

Sol^{no} - $-(x+1)(x^2-b^2) = 0$

$$x = -1, b, -b$$

now, $\frac{d^2x}{dt^2} = -1 \left((x^2-b^2) + (x+1)2x \right)$

For stable points: $\frac{d^2x}{dt^2} > 0$

$$\text{at } x=-1: \frac{d^2x}{dt^2} = -[1-b^2] = b^2-1 \quad \text{--- (1)}$$

$$\text{at } x=b: \frac{d^2x}{dt^2} = -[0+2b(b+1)] = -2b(b+1) \quad \text{--- (2)}$$

$$\text{at } x=-b: \frac{d^2x}{dt^2} = -[0-2b(-b+1)] = 2b(1-b) \quad \text{--- (3)}$$

from eqn (2) & (3)

$$b > 1$$

[B]

Ques A particle of mass m is moving in the potential $V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$

The frequency of small oscillation about the state equilibrium is:

a) $\sqrt{\frac{a}{m}}$ b) $\sqrt{\frac{2a}{m}}$ c) $\sqrt{\frac{3a}{m}}$ d) $\sqrt{\frac{6a}{m}}$

Solⁿo- $V'(x) = -ax + bx^3 = 0$

$$x = 0, \pm \sqrt{\frac{a}{b}}$$

$$V''(x) = -a + 3bx^2$$

$$V''(x) \Big|_{x=\pm\sqrt{\frac{a}{b}}} = 2a$$

$$\omega = \sqrt{\frac{V''(x)}{m}} = \sqrt{\frac{2a}{m}}$$

[B]

Que 6 A particle of mass m moves in the one-⁽³⁰⁾
dimensional potential:

$$V(x) = \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4$$

one of the equilibrium points is $x=0$. The
angular frequency of small oscillation about the
other eqm point is:

a) $\frac{2\alpha}{\sqrt{3m\beta}}$ b) $\frac{\alpha}{\sqrt{m\beta}}$ c) $\frac{\alpha}{\sqrt{12m\beta}}$ d) $\frac{\alpha}{\sqrt{24m\beta}}$

Solⁿo $V'(x) = \alpha x^2 + \beta x^3 = 0$

$$x^2(\alpha + \beta x) = 0$$

$$x = -\alpha/\beta, 0, 0$$

$$V''(x) = 2\alpha x + 3\beta x^2$$

$$x = -\alpha/\beta \quad = 2\alpha\left(-\frac{\alpha}{\beta}\right) + 3\beta\left(-\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2}{\beta}$$

$$\omega = \sqrt{\frac{\alpha^2/\beta}{m}} = \frac{\alpha}{\sqrt{m\beta}} \quad \boxed{B}$$

Que 7 For a dynamical system governed by equation

$$\frac{dx}{dt} = 2\sqrt{1-x^2} \quad \text{with } |x| \leq 1$$

a) $x=-1$ & $x=1$ are both unstable fixed points

b) $x = -1$ & $x = 1$ are both stable points (3)

c) $x = -1$ is an unstable fixed point & $x = 1$ is stable fixed point.

d) $x = -1$ is stable fixed point & $x = 1$ is an unstable fixed point.

solⁿo- for fix points:

$$\frac{dx}{dt} = 0 = 2\sqrt{1-x^2}$$

$$\Rightarrow x = \pm 1$$

$$\frac{d^2x}{dt^2} = 2x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} (-2x) = \frac{-2x}{\sqrt{1-x^2}}$$

$$\left. \frac{d^2x}{dt^2} \right|_{x=1} = \frac{-2}{\sqrt{\frac{1}{1}-1}} =$$

RADIAL SMALL OSCILLATION

(32)

$$\omega_r = \sqrt{\frac{V''_{\text{eff}}}{m}} \quad \text{where} \quad V_{\text{eff}}(r) = \frac{J^2}{2mr^2} + V(r)$$

$$T = \begin{pmatrix} a & c \\ c & b \end{pmatrix}_{2 \times 2} \quad V = \begin{pmatrix} d & g \\ g & f \end{pmatrix}_{2 \times 2}$$

$$T = ax_1^2 + bx_2^2 + 2cx_1x_2$$

$$V = dx_1^2 + fx_2^2 + 2gx_1x_2$$

For normal frequency:

$$\boxed{|V - \omega^2 T| = 0}$$

↑
normal frequency

For normal modes:

$$\boxed{(V - \omega^2 T) |\omega_i\rangle = |0\rangle}$$

↑
normal mode.

→ Brachistochrone:

$$T_{\text{min}} = \pi \sqrt{\frac{l}{g}}$$

→ Huygens Pendulum

$$T = 4\pi \sqrt{\frac{l}{g}}$$

For cycloid.

Ques 1 A planet of mass m & angular momentum L moves in a circular orbit in a potential $V(r) = -k/r$. If it is slightly perturbed radially, the angular frequency of radial oscillation is:

$$a) \frac{mk^2}{\sqrt{2}L^3} \quad b) \frac{mk^2}{L^3} \quad c) \frac{\sqrt{2}mk^2}{L^3} \quad d) \frac{\sqrt{3}mk^2}{L^3}$$

Sol^{no} -

$$E_{\text{total}} = K.E + P.E$$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{K}{r}}_{\uparrow V_{\text{eff}}}$$

$$\frac{dV_{\text{eff}}}{dr} = -\frac{L^2}{mr^3} + \frac{K}{r^2} = 0 \Rightarrow r_0 = \frac{L^2}{mK}$$

$$\begin{aligned} \frac{d^2V_{\text{eff}}}{dr^2} &= \frac{3L^2}{mr^4} - \frac{2K}{r^3} \Rightarrow \frac{1}{r_0^3} \left[\frac{3L^2}{mr_0} - 2K \right] \\ &= \frac{m^3 K^4}{L^6} \end{aligned}$$

$$\omega = \sqrt{\frac{d^2V_{\text{eff}}/dr^2}{m}} = \frac{mK^2}{L^3} \quad \boxed{B}$$

Ques The Lagrangian of a system is:

$$L = \frac{1}{2} m \dot{q}_1^2 + 2m \dot{q}_2^2 - K \left(\frac{5}{4} q_1^2 + 2q_2^2 - 2q_1 q_2 \right)$$

The frequencies of its normal modes are:

a) $\sqrt{\frac{K}{2m}}, \sqrt{\frac{3K}{m}}$ b) $\sqrt{\frac{K}{2m}} (1 \pm \sqrt{3})$

c) $\sqrt{\frac{5K}{2m}}, \sqrt{\frac{K}{m}}$ d) $\sqrt{\frac{K}{2m}}, \sqrt{\frac{6K}{m}}$

Sol^{no} -

$$L = \frac{1}{2} m \dot{q}_1^2 + \frac{4m}{2} \dot{q}_2^2 - \frac{K}{2} \left(\frac{10}{4} q_1^2 + 4q_2^2 - 2q_1 q_2 - 2q_1 q_2 \right)$$

$$T = \begin{pmatrix} m & 0 \\ 0 & 4m \end{pmatrix}; \quad V = \begin{pmatrix} \frac{10}{4} K & -2K \\ -2K & 4K \end{pmatrix}$$

$$|V - \omega^2 r| = 0$$

(34)

$$\begin{vmatrix} \frac{10}{4}K - \omega^2 m & -2K \\ -2K & 4K - 4\omega^2 m \end{vmatrix} = 0$$

$$4 \left(\frac{10}{4}K - \omega^2 m \right) (K - \omega^2 m) - 4K^2 = 0$$

$$4 [10K^2 - 10\omega^2 K m - 4\omega^2 K m + 4\omega^4 m^2] - 4K^2 = 0$$

on simplification:

$$\omega = \sqrt{\frac{K}{2m}} \text{ \& \ } \sqrt{\frac{3K}{m}} \quad \boxed{A}$$

CENTRAL FORCE

- Torque: zero
- angular momentum: conserved.
- motion: confined in plane
- E_{total} : conserved
- Axial velocity: constant.

→ $r^n = a \cos n\theta$ then

$$f(r) \propto \frac{1}{r^{2n+3}}$$

Energy: $E = -\frac{mK^2}{2L^2}$ for circle.

→ eccentricity (e):

$e=0$ circle

$e=1$ Parabola

$e > 1$ Hyperbola

$0 < e < 1$ ellipse.

velocity

Path

35

$$\sqrt{\frac{\mu M}{R+r}} \longrightarrow \text{circle}$$

$$\sqrt{\frac{2\mu M}{R+r}} \longrightarrow \text{Parabola}$$

$$\text{Energy} = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{J^2}{m r^2}}_{V_{\text{eff}}} + V(r)$$

law of Period: $T^2 \propto a^3$

PHASE SPACE TRAJECTORY

(36)

Phase space = coordinate space + momentum space

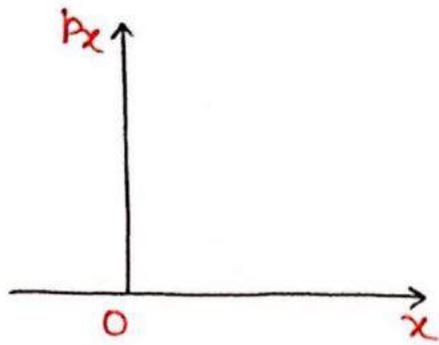


Fig: Phase space

Dimension of Phase Space = $2 \times \text{DOF}$

Equation of PST: If potential doesn't depend on velocity term then

$$H \equiv E = \text{constant of motion}$$

$$H = \frac{p_x^2}{2m} + V(x)$$

Properties:

- 1) For time independent system ($H = \text{constant of motion}$) PST's don't intersect each other.
- 2) For bounded motion, PST are closed curves.
- 3) arrow shows the sign of momentum
 $p = +ve$: arrow is towards +ve x-axis
 $p = -ve$: arrow is towards -ve x-axis
- 4) If Hamiltonian contains only even powers of p_x then PST will be symmetric about x-axis.

5) If one variable in 2nd power & other in 1st power i.e. (37)

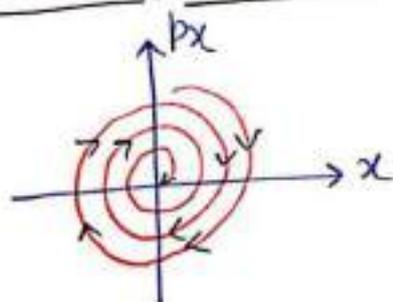
$$E = \frac{kx^2}{2m} + mgx$$

then shape is Parabolic.

For Damped (Dissipative) cases:

similar to PST of undamped cases except that all PST will be connected together like spiral.

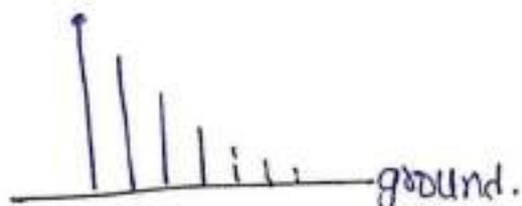
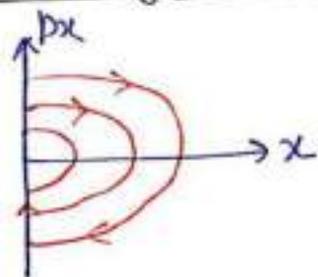
1) For Damped Harmonic Oscillator:



e.g. Pendulum set into motion by giving impulse.

E goes on \downarrow ing.

2) Ball Bouncing inelastically:



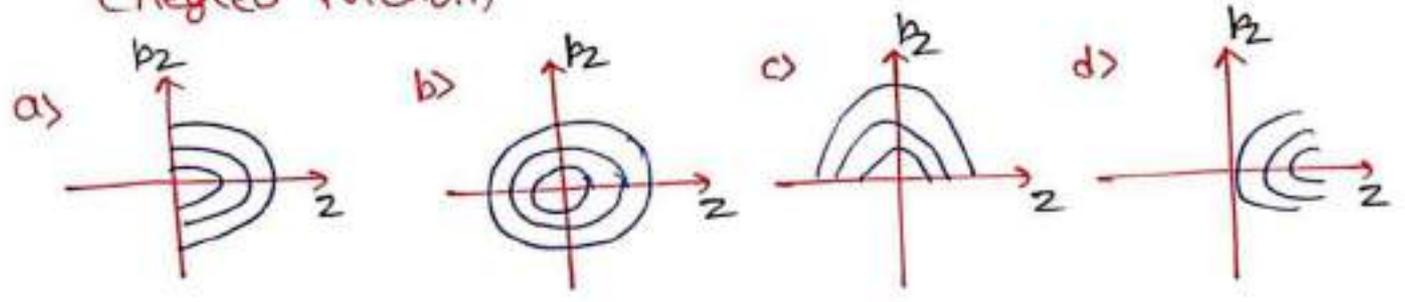
How to solve Problems:

Step I: Put $V(x) = 0$ --- Find: x_1, x_2

Step II: Put $V'(x) = 0$ --- Find: x_0

Step III: find local minima & maxima.

Ques! The trajectory on the z - p_z -plane (Phase-space trajectory) of a ball bouncing perfectly elastically off a hard surface at $z=0$ is given by approximate (neglect friction)



Solⁿ:- $E = \frac{p_z^2}{2m} + mgz$ $V(x) = mgz$

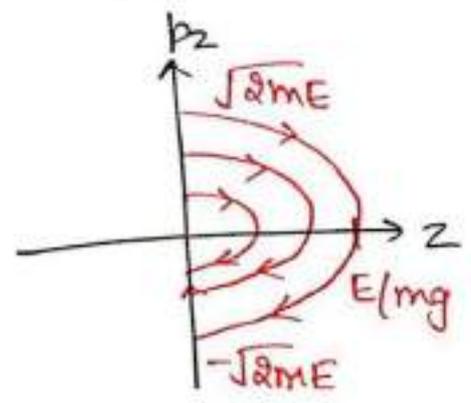
when $z=0$

$$p_z^2 = 2mE \Rightarrow p_z = \pm \sqrt{2mE}$$

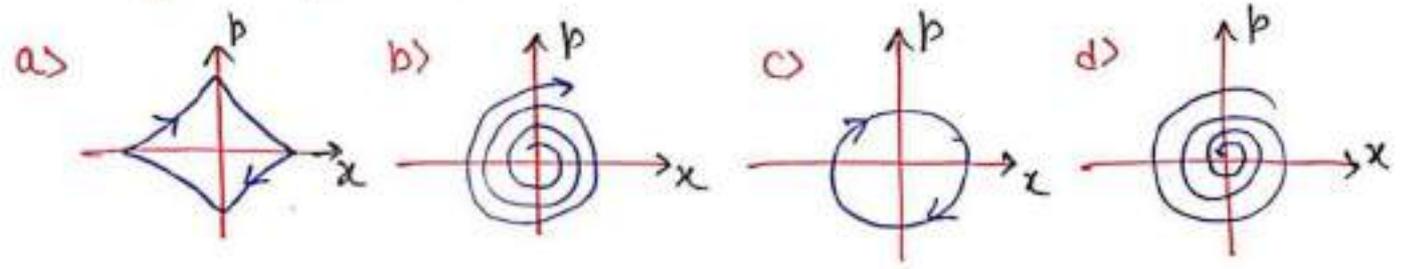
when $p_z=0$

$$z = E/mg$$

[A]

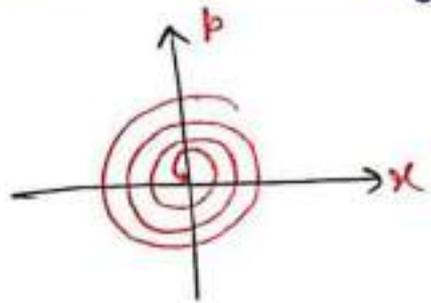


Ques! The bob of a simple pendulum, which undergoes small oscillations, is immersed in water. which of the following figures best represents the phase diagram for the pendulum?



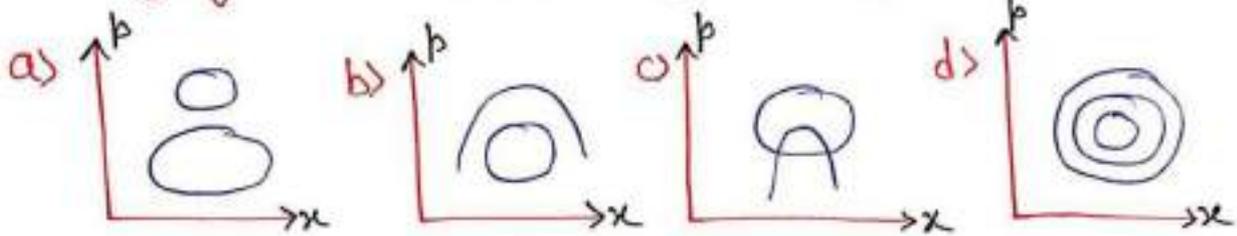
Sol^{no}:- when simple pendulum oscillates in water damped oscillations are produced.

i.e amplitude continuously ↓ & finally stops



□

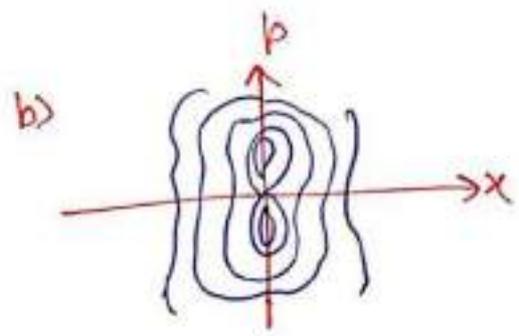
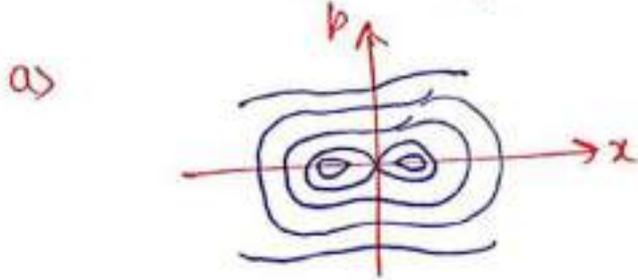
Que³ which of the following set of phase space trajectories which one is not possible for a particle obeying Hamiltonian equation of motion?

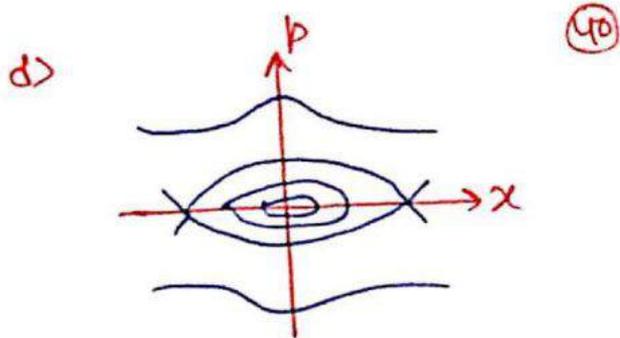
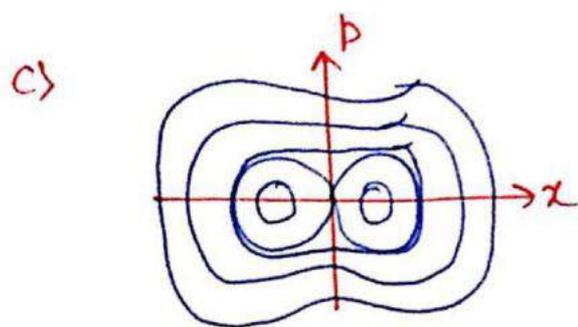


Sol^{no}:- Phase space curves don't intersect each other when Hamiltonian is constant. □

Que⁴ which of the following figures is a schematic representation of the PST (i.e contour of constant energy) of a particle moving in a 1-Dim. potential

$$V(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3$$





(40)

Sol^{no} - For constant energy:

→ PST don't intersect each other

Ⓒ & Ⓓ eliminated.

now put $x=0$

$$E = \frac{p^2}{2m} - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

$$p = \pm \sqrt{2mE} \rightarrow \boxed{A}$$

for $x \rightarrow \infty$

$$E \rightarrow \infty \quad px \rightarrow 0$$

Ques 5 The Lagrangian of a free relativistic particle (in 1-dimension) of mass m is given by:

$$L = -m\sqrt{1-\dot{x}^2} \quad \text{where } \dot{x} = dx/dt$$

If such a particle is acted upon by a constant force in the direction of its motion, the PST obtained from the corresponding Hamiltonian are:

a) ellipse

b) cycloid

c) hyperbola

d) Parabola.

Solⁿo-

$$L = T - V = -m\sqrt{1-\dot{x}^2} - V(x)$$

(4)

$$H = \dot{q} p_x - L$$

$$\frac{\partial L}{\partial \dot{x}} = p_x = \frac{+m \cdot 2\dot{x}}{2\sqrt{1-\dot{x}^2}} = \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}} \quad \text{--- (a)}$$

$$p_x^2 (1-\dot{x}^2) = m^2 \dot{x}^2$$

$$p_x^2 = (p_x^2 + m^2) \dot{x}^2$$

$$\frac{p_x}{\dot{x}} = \sqrt{p_x^2 + m^2} \quad \text{--- (1)}$$

$$H = \dot{x} \cdot \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}} + m\sqrt{1-\dot{x}^2} - V(x)$$

$$= \frac{m\dot{x}^2 + m - m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - V(x)$$

$$= \frac{m}{\sqrt{1-\dot{x}^2}} - V(x)$$

$$= \frac{p_x}{\dot{x}} - V(x) \quad \text{By eqn (a)}$$

$$= \sqrt{p_x^2 + m^2} - V(x) \quad \text{by eqn (1)}$$

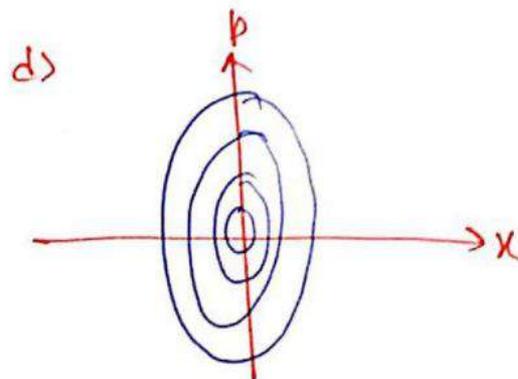
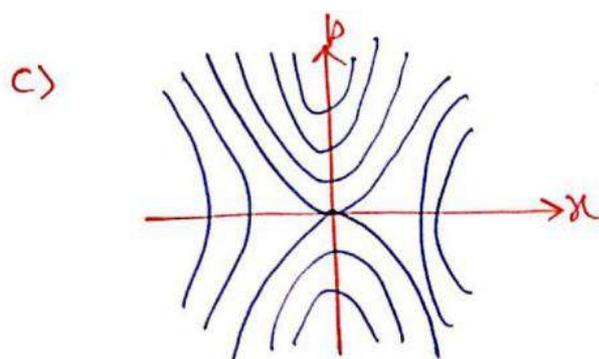
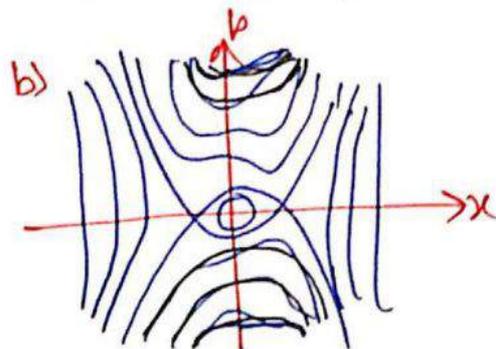
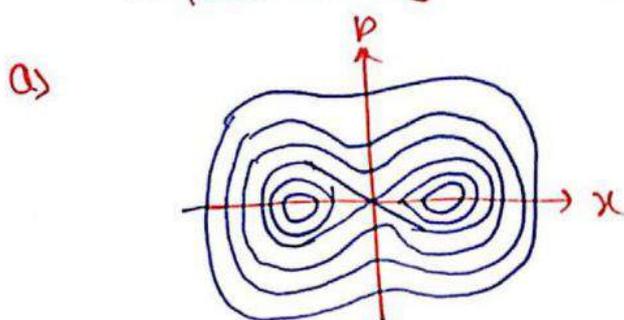
$$(E + V(x)) = \sqrt{p_x^2 + m^2}$$

$$\boxed{(E + V(x))^2 - p_x^2 = m^2} \quad \text{hyperbola. } \boxed{c}$$

Ques 6 (4) A particle moves in one dimension in potential

$$V(x) = -k^2 x^4 + \omega^2 x^2$$

which of the following curves best describes the trajectories of this system in phase space?



Sol^{no}:- $E = \frac{p^2}{2m} - k^2 x^4 + \omega^2 x^2$

when $x=0$

$$E = \text{constant} = \frac{p^2}{2m} \rightarrow \textcircled{a} \textcircled{b} \textcircled{d} \text{ eliminated.}$$

exact method:

$$V(x)=0 : k^2 x^4 = \omega^2 x^2$$

$$x = \pm \omega/k, 0, 0$$

$$V'(x) = -4k^2 x^3 + 2\omega^2 x = 0$$

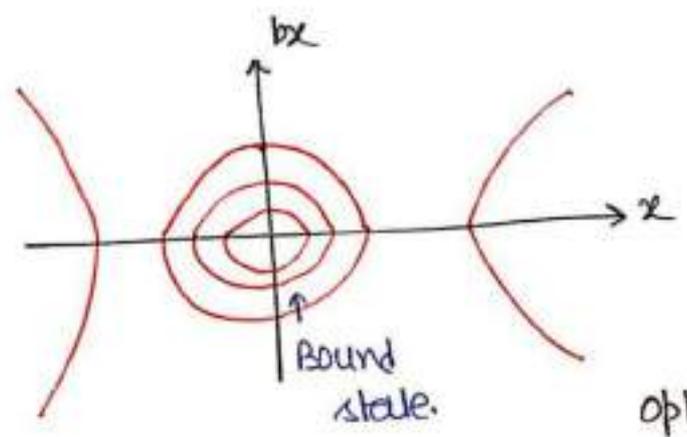
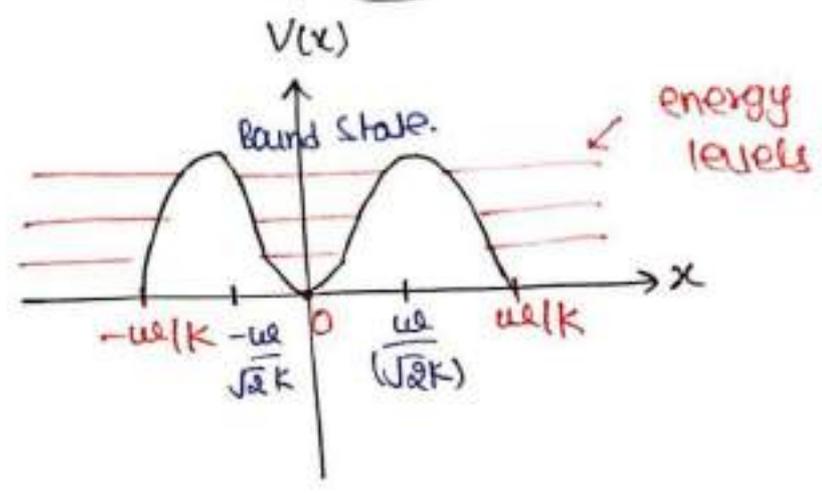
$$x(2\omega^2 - 4k^2 x^2) = 0$$

$$x = 0, \pm \frac{\omega}{\sqrt{2}k}$$

$$V''(x) = -12k^2x^2 + 2ue^2$$

$$V''(x) \Big|_{x=0} = 2ue^2 \text{ (true) minima (x=0)}$$

$$x''(x) \Big|_{x = \pm \frac{ue}{\sqrt{2}k}} = -12k^2 \frac{ue^2}{2k^2} + 2ue^2 = -4ue^2 \text{ maxima. (x = } \pm \frac{ue}{\sqrt{2}k} \text{)}$$



only option **B**

SPECIAL THEORY OF RELATIVITY

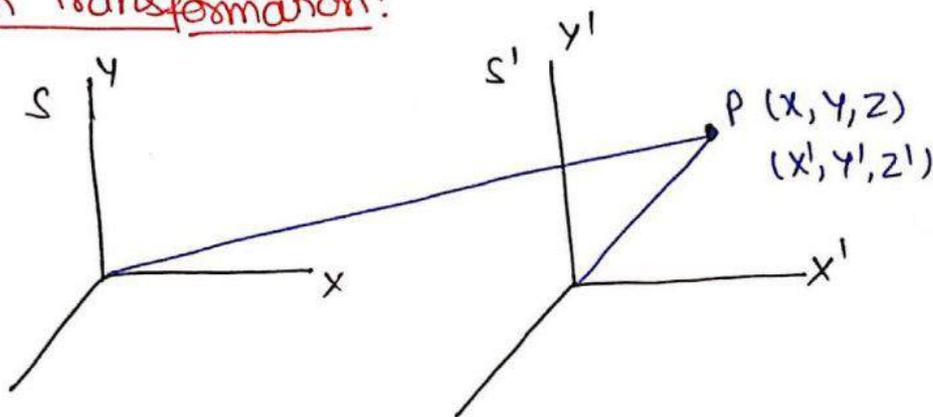
(44)

originated from the analysis of Maxwell's equation (electromagnetic theory)

Postulates:

- all inertial frames are equivalent
- speed of light in vacuum is invariant in all inertial frames.

Galilean Transformation:



$$\begin{array}{l|l|l} x' = x - vt & v_x' = v_x - v & a_x' = a_x \\ y' = y & v_y' = v_y & a_y' = a_y \\ z' = z & v_z' = v_z & a_z' = a_z \\ t' = t & & \end{array} \quad \begin{array}{l} \text{Thus, acceleration} \\ \text{is invariant} \\ \text{under G.T} \end{array}$$

→ length of object: $\Delta x' = \Delta x$
is invariant under G.T

→ let P be a Photon

$$v_x = c \quad \text{--- (1)}$$

$$\text{acc. to G.T: } v_x' = v_x - v = c - v$$

$v_x' \neq v_x$ → failure of G.T

Lorentz Transformation:

(45)

Einstein modified the LT to make it compatible with postulates of STR.

$$\begin{aligned}x' &= \gamma (x - vt) \\y' &= y \\z' &= z \\t' &= \gamma \left(t - \frac{xv}{c^2} \right)\end{aligned}$$

By Postulates of STR:

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\Delta x' = \frac{\Delta x - v \Delta t}{\sqrt{1 - v^2/c^2}}$$

$$\Delta y' = \Delta y, \quad \Delta z' = \Delta z$$

$$\Delta t' = \frac{\Delta t - \frac{v \Delta x}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Differential form of Lorentz transformation.

Inverse Lorentz Transformation:

If values are known in S' frame & values in S frame are found then we use inverse L.T.

$$x = \gamma (x' + vt')$$

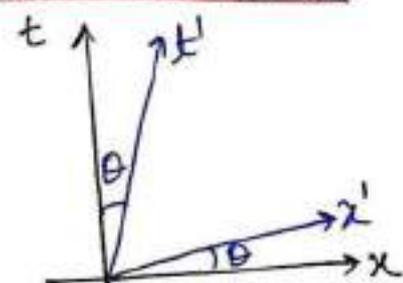
$$t = \gamma \left(t' + \frac{x'v}{c^2} \right)$$

Lorentz Transformation for fictitious rotation in space time (x & t)

$$\text{let } \frac{v}{c} = \tan R\theta$$

$$x' = x \cos R\theta - ct \sin R\theta$$

$$ct' = ct \cos R\theta - x \sin R\theta$$



velocity Transformation:

(46)

$$v_x' = \frac{dx'}{dt'} = \frac{dx - v dt}{dt - \frac{v dx}{c^2}} = \frac{\left(\frac{dx}{dt} - v\right) dt}{\left(1 - \frac{v}{c^2} \frac{dx}{dt}\right) dt}$$

$$v_x' = \frac{v_x - v}{1 - \frac{v v_x}{c^2}}$$

$$v_x = \frac{v_x' + v}{1 + \frac{v v_x'}{c^2}}$$

$$v_y' = \frac{v_y}{\gamma \left(1 - \frac{v v_x}{c^2}\right)}$$

$$v_y = \frac{v_y'}{\gamma \left(1 + \frac{v v_x'}{c^2}\right)}$$

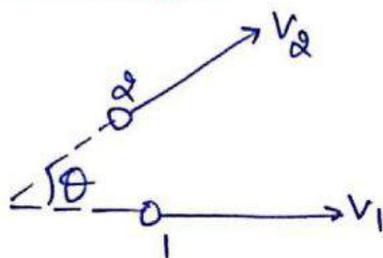
$$v_z' = \frac{v_z}{\gamma \left(1 - \frac{v v_x}{c^2}\right)}$$

$$v_z = \frac{v_z'}{\gamma \left(1 + \frac{v v_x'}{c^2}\right)}$$

↑

inverse velocity transformation.

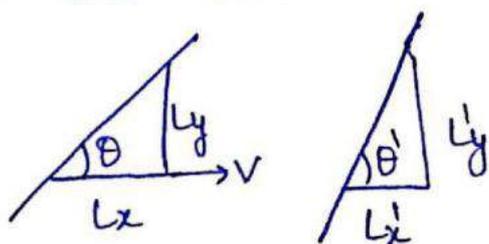
oblique Velocity Transformation:



$$\vec{v}_{21} = -\vec{v}_{12}$$

$$v_{21} = \frac{\sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta - \frac{v_1^2 v_2^2 \sin^2 \theta}{c^2}}}{1 - \frac{v_1 v_2 \cos \theta}{c^2}}$$

Relativity in geometrical angle:



$$\tan \theta' = \frac{L_y'}{L_x'}$$

$$\tan \theta = \frac{L_y}{L_x}$$

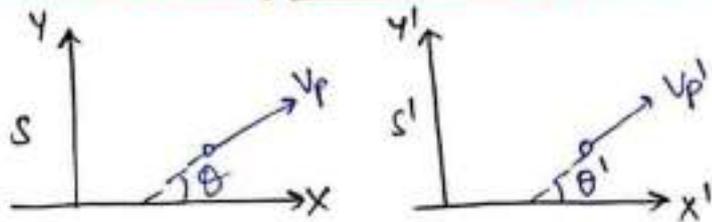
$$L_x' = L_x \sqrt{1 - v^2/c^2} \rightarrow \text{By length contraction.}$$

$$L_y' = L_y$$

$$\tan \theta' = \frac{L_y}{L_x \sqrt{1 - v^2/c^2}} = \frac{\tan \theta}{\sqrt{1 - v^2/c^2}}$$

$$\tan \theta' = \frac{\tan \theta}{\sqrt{1 - v^2/c^2}} \quad \theta' > \theta.$$

Relativity of angle made by velocity vector:

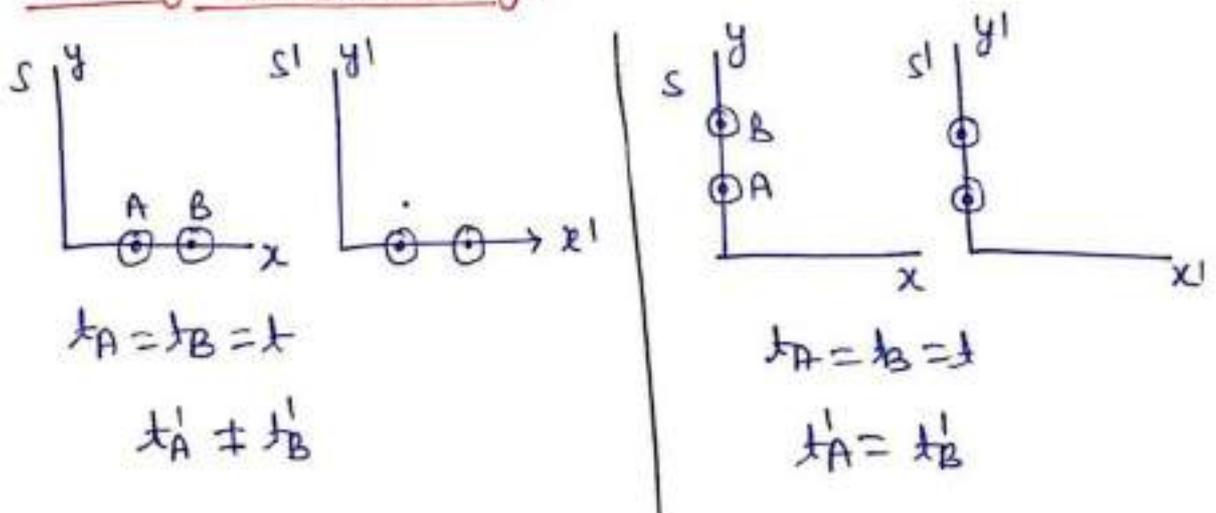


$$v_p = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad \& \quad v_{p'} = \sqrt{v_{x'}^2 + v_{y'}^2 + v_{z'}^2}$$

$$\cos \theta = \frac{v_x}{v_p} \quad \& \quad \cos \theta' = \frac{v_{x'}}{v_{p'}}$$

Consequence of Relativity:

1) loss of simultaneity:

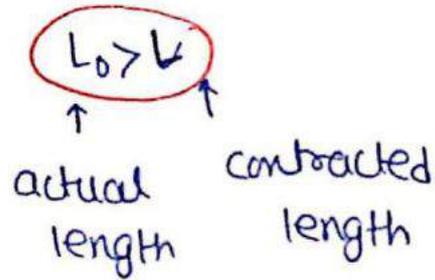


2) Length contraction:

48

$$L = \frac{L_0}{\gamma} = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

similarly.



$$A = A_0 \sqrt{1 - \frac{v^2}{c^2}} \longrightarrow \text{area contraction}$$

$$V = V_0 \sqrt{1 - \frac{v^2}{c^2}} \longrightarrow \text{volume contraction}$$

3) Time Dilation:

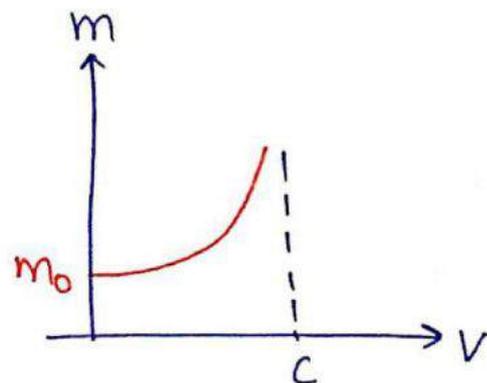
means slowing down of a process while in motion.

$$t = t_0 \gamma = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t > t_0$$

4) Relativity in mass:

$$m = m_0 \gamma = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$



5) mass energy equivalence:

greater the energy greater will be the mass and vice-versa.

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

quantity	Relativistic	Non-Relativistic
Momentum	$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$	$p = m_0 v$
Kinetic Energy	$K = mc^2 - m_0 c^2$	$K = \frac{1}{2} m_0 v^2$
Total energy	$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$	$E = m_0 c^2$
K.E & momentum Relation	$pc = \sqrt{K(K + 2m_0 c^2)}$	$K = \frac{p^2}{2m_0}$
energy-momentum Relation	$E = \sqrt{p^2 c^2 + m_0^2 c^4}$	$E = \frac{p^2}{2m_0}$

Loventz Invariant quantities:

- Rest mass (m_0), Proper time (t_0), proper length (l_0)
- $E^2 - p^2 c^2$; $E^2 - B^2 c^2$; $\vec{E} \cdot \vec{B}$; $|\vec{E}| = |\vec{B}|$
- $x^2 + y^2 + z^2 - c^2 t^2$; $\Delta r^2 - c^2 \Delta t^2$
- Phase of wave i.e $\vec{k} \cdot \vec{r} - \omega t$
- Maxwell equation
- equation of wave

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \quad \text{OR} \quad \nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

- $\mu_0 \epsilon_0 = 1/c^2$
- charge

Energy - momentum Transformation:

(50)

$$E = \gamma m_0 c^2 \times \frac{\Delta t}{\Delta t}$$

$$\frac{\Delta t}{\gamma} = \Delta t_0$$

$$E = \left(\frac{m_0 c^2}{\Delta t_0} \right) \Delta t$$

Lorentz invariant

$$p_x = \left(\frac{m_0}{\Delta t_0} \right) \Delta x$$

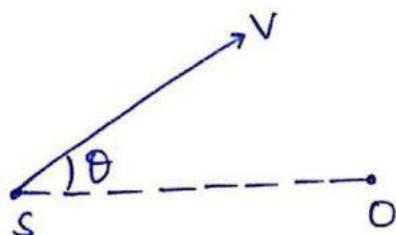
→ E should transform in same way as Δt does
→ similarly, p should transform in same way as Δx does.

$$p'_x = \frac{p_x - \frac{vE}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$E' = \frac{E - vp_x}{\sqrt{1 - v^2/c^2}}$$

Relativistic Doppler Effect:

change in apparent frequency of an em wave due to relative motion between source and observer is called Doppler effect.



$$\nu = \frac{\nu_0 \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c} \cos \theta}$$

ν_0 → apparent frequency of em wave

ν_0 → actual frequency of em wave

v → relative velocity b/w source & observer.

Cases:

1) Source & observer moving along line ($\theta=0$)

$$v = v_0 \sqrt{\frac{1+v/c}{1-v/c}} \quad \& \quad \lambda = \lambda_0 \sqrt{\frac{1-v/c}{1+v/c}}$$

2) $\theta=180^\circ$. (Distance ↑ as blue s40)

$$v = v_0 \sqrt{\frac{1-v/c}{1+v/c}} \quad \& \quad \lambda = \lambda_0 \sqrt{\frac{1+v/c}{1-v/c}}$$

3) If v is \perp to line joining s40 ($\theta=90^\circ$).

$$v = v_0 \sqrt{1 - \frac{v^2}{c^2}}$$

Newton's law in Relativity:

Newton's second law: $\vec{F} = \frac{d\vec{p}}{dt}$

$$\vec{F} = \frac{d}{dt}(m_0 \vec{v}) = m_0 \frac{d}{dt} \left[\frac{v}{\sqrt{1-v^2/c^2}} \right]$$

$$\vec{F} = \frac{m_0 \frac{dv}{dt}}{\sqrt{1-v^2/c^2}} + \frac{m_0 (v \cdot \frac{dv}{dt}) \frac{v}{c^2}}{(1-v^2/c^2)^{3/2}}$$

$$\vec{F} = \frac{m_0 \vec{a}}{\sqrt{1-v^2/c^2}} + \frac{m_0 (\vec{v} \cdot \vec{a}) v / c^2}{(1-v^2/c^2)^{3/2}}$$

when $\vec{v} \parallel \vec{a}$ are \parallel :

$$F = \frac{m_0 a}{(1-v^2/c^2)^{3/2}}$$

when $\vec{v} \perp \vec{a}$:

$$F = \frac{m_0 a}{(1-v^2/c^2)^{1/2}}$$

Transformation of Force: (52)

$$F_x' = \frac{dp_x'}{dt'} = \frac{\gamma(dx - v \frac{dE}{c^2})}{\gamma(dt - \frac{v dx}{c^2})} = \frac{\frac{dp_x}{dt} - \frac{v}{c^2} \frac{dE}{dt}}{1 - \frac{v}{c^2} \frac{dx}{dt}}$$

$$F_x' = \frac{F_x - \frac{v}{c^2} (\vec{F} \cdot \vec{v}_p)}{1 - \frac{v v_x}{c^2}}$$

$$F_y' = \frac{F_y}{\gamma(1 - \frac{v v_x}{c^2})}$$

$$F_z' = \frac{F_z}{\gamma(1 - \frac{v v_x}{c^2})}$$

For instantaneous rest frame:

$$v_p = 0$$

$$F_x' = F_x$$

$$F_y' = F_y / \gamma$$

$$F_z' = F_z / \gamma$$

Classification of interval between two events:

1) Space like:

$$\text{space time interval: } ds^2 = \Delta r^2 - c^2 \Delta t^2$$

If $\Delta r > c \Delta t \rightarrow$ space like

two events can't occur simultaneously

2) Time like:

$$\Delta r < c \Delta t \rightarrow \text{time like}$$

two events can't occur simultaneously

3) light like:

$$\Delta r = c \Delta t \rightarrow \text{light like.}$$

two events can causally connect.

Ques! A constant force F is applied to a relativistic particle of rest mass m . If the particle starts from rest at $t=0$, its speed after a time t is: (53)

- a) $\frac{Ft}{m}$ b) $c \tanh\left(\frac{Ft}{mc}\right)$ c) $c(1 - e^{-Ft/mc})$ d) $\frac{Fct}{\sqrt{F^2t^2 + m^2c^2}}$

Solⁿo-

$$F = \frac{dp}{dt}$$

$m \rightarrow$ Rest mass

$$dp = F dt \Rightarrow p = Ft + \text{constant} \quad \text{--- (1)}$$

$$p = \gamma m v = \frac{mv}{\sqrt{1 - v^2/c^2}} \quad \text{--- (2)}$$

From eqn (1) & (2)

$$F^2 t^2 = \frac{m^2 v^2}{1 - v^2/c^2}$$

$$1 - \frac{v^2}{c^2} = \frac{m^2 v^2}{F^2 t^2}$$

$$1 = \frac{m^2 v^2}{F^2 t^2} + \frac{v^2}{c^2} \Rightarrow v^2 \left[\frac{m^2 c^2 + F^2 t^2}{F^2 c^2 t^2} \right]$$

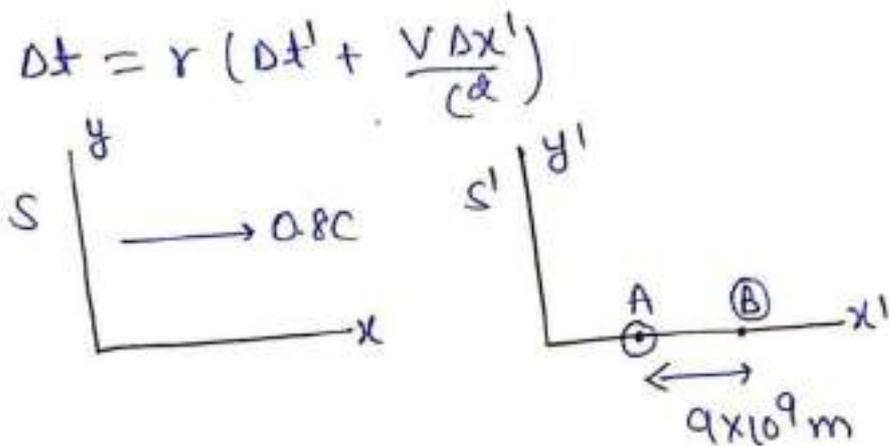
$$v^2 = \frac{F^2 c^2 t^2}{F^2 t^2 + m^2 c^2}$$

$$v = \frac{Fct}{\sqrt{F^2 t^2 + m^2 c^2}}$$

D

Ques 2 (54)
 Two events are separated by a (spatial) distance $9 \times 10^9 \text{ m}$, are simultaneous in one inertial frame. The time interval between these two events in a frame moving with constant speed $0.8c$ is:
 a) 60 s b) 40 s c) 20 s d) 0 s

Solⁿg



given: $\Delta t' = 0$, $\Delta x' = 9 \times 10^9 \text{ m}$, $v = 0.8c$

$$\begin{aligned} \Delta t &= \frac{1}{\sqrt{1 - (0.8)^2}} \left(0 + \frac{0.8c}{c^2} \times 9 \times 10^9 \right) \\ &= \frac{1}{0.6} \left(\frac{0.8}{3 \times 10^8} \times 9 \times 10^9 \right) = 40 \text{ sec} \end{aligned}$$

B

Ques 3 what is the proper time interval between the occurrence of two events, if in one inertial frame the events are separated by $7.5 \times 10^5 \text{ km}$ & occurs 6.5 s apart?

a) 6.5 s b) 6.0 s c) 5.75 s d) 5.0 s

Solⁿ:- $\Delta t_0 = ?$

$$\Delta x' = 7.5 \times 10^5 \text{ Km} = 7.5 \times 10^8 \text{ m}$$

$$\Delta t' = 6.5 \text{ s.}$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \Rightarrow \Delta t_0 = \Delta t \sqrt{1 - v^2/c^2}$$

$$\Delta t_0^2 = \Delta t^2 - \frac{v^2 \Delta t^2}{c^2} \quad (\because v \Delta t = \Delta x)$$

$$= (6.5)^2 - \left(\frac{7.5 \times 10^8}{3 \times 10^8} \right)^2$$

$$= 42.25 - 6.25 = 36 \text{ sec}^2$$

$$\Delta t_0 = 6 \text{ sec}$$

B

Ques 4) which one of the following quantities are Lorentz invariant?

a) $|E \times B|^2$ b) $|E|^2 - |B|^2$ c) $|E|^2 + |B|^2$ d) $|E||B|^2$

Solⁿ:- $|E|^2 - |B|^2 = |E'|^2 - |B'|^2$

↑ Lorentz invariant.

B

Ques 5) Let v, p & E denotes the speed, the magnitude of the momentum & the energy of a free particle of rest mass (m). Then.

a) $\frac{dE}{dp} = \text{constant}$ b) $p = mv$ c) $v = \frac{cp}{\sqrt{p^2 + m^2 c^2}}$ d) $E = mc^2$

Solⁿ:- $E = \sqrt{m^2 c^4 + p^2 c^2}$ $m \rightarrow$ Rest mass (56)

velocity of Particle (v) = V_g

$$V = \frac{dE}{dp} = \frac{1}{2} \times \frac{1}{\sqrt{m^2 c^4 + p^2 c^2}} \times 2pc^2$$

$$= \frac{pc^2}{c \sqrt{p^2 + m^2 c^2}} = \frac{pc}{\sqrt{p^2 + m^2 c^2}} \quad \square$$

alternate:

$$E = \gamma m c^2 = \sqrt{p^2 c^2 + m^2 c^4} \quad ; \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$V = \frac{pc}{\sqrt{p^2 + m^2 c^2}}$$

Que 6 The area of a disc in its rest frame S is equal to 1 (in some units). The disc will appear distorted to an observer O moving with a speed u w.r.t S along the plane of the disc. The area of the disc measured in the rest frame of the observer O is:

- a) $(1 - \frac{u^2}{c^2})^{1/2}$ b) $(1 - \frac{u^2}{c^2})$ c) $(1 - \frac{u^2}{c^2})$ d) $(1 - \frac{u^2}{c^2})^{-1}$

Solⁿ:- $A = A_0 \sqrt{1 - \frac{v^2}{c^2}}$ here $v = u$.

$$\frac{A}{A_0} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

A

51

Ques 7 A light source is switched on and off at constant frequency f . An observer moving with a velocity u w.r.t the light source will observe the frequency of the switching to be:

a) $f \left(1 - \frac{u^2}{c^2}\right)^{-1}$

b) $f \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$

c) $f \left(1 - \frac{u^2}{c^2}\right)$

d) $f \left(1 - \frac{u^2}{c^2}\right)^{1/2}$

Solⁿo- $t' = \frac{t}{\sqrt{1 - v^2/c^2}} \rightarrow$ time dilation.

$$\frac{1}{f'} = \frac{1}{f} \sqrt{1 - v^2/c^2}$$

$$f' = f \left(1 - \frac{u^2}{c^2}\right)^{1/2} \quad \boxed{D}$$

Ques 8 According to STR, the speed v of a free particle of mass m and total energy E is:

a) $v = c \sqrt{1 - \frac{mc^2}{E}}$

b) $v = \sqrt{\frac{2E}{m}} \left(1 + \frac{mc^2}{E}\right)$

c) $v = c \sqrt{1 - \left(\frac{mc^2}{E}\right)^2}$

d) $v = c \left(1 + \frac{mc^2}{E}\right)$

Solⁿo-

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$1 - \frac{v^2}{c^2} = \frac{m^2 c^4}{E^2}$$

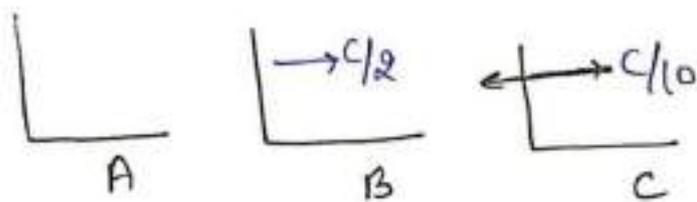
$$\frac{v^2}{c^2} = 1 - \frac{m^2 c^4}{E^2}$$

$$v = c \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} \quad \boxed{C}$$

Ques 9 Consider three inertial frames of reference A, B & C. The frame B moves with a velocity $c/2$ w.r.t A and C moves with a velocity $c/10$ w.r.t B in the same direction. The velocity of C as measured in A is:

- a) $\frac{3c}{7}$ b) $\frac{4c}{7}$ c) $\frac{c}{7}$ d) $\frac{\sqrt{3}}{7}c$

Solⁿ



$$v = \frac{v_{CB} + v_{BC}}{1 + \frac{v_{CB} v_{BC}}{c^2}} = \frac{\frac{c}{2} + \frac{c}{10}}{1 + \frac{1}{2} \times \frac{1}{10}} = \frac{4}{7}c \quad \boxed{B}$$

Ques 10 A rod of length L carries a total charge Q distributed uniformly. If this is observed in a frame

moving with speed v along the rod, charge per unit length is: (50)

a) $\frac{Q}{L} \left(1 - \frac{v^2}{c^2}\right)$ b) $\frac{Q}{L} \sqrt{1 - \frac{v^2}{c^2}}$ c) $\frac{Q}{L \sqrt{1 - \frac{v^2}{c^2}}}$ d) $\frac{Q}{L \left(1 - \frac{v^2}{c^2}\right)}$

Sol^{no} Q: Lorentz Invariant

$$\frac{Q}{L} = \frac{Q}{L_0 \sqrt{1 - v^2/c^2}} \quad \boxed{c}$$

Ques 11 Consider a particle of mass m moving with a speed v . If T_R denotes the relativistic kinetic energy & T_N denotes the non-relativistic K.E approximation, then the value of $\left(\frac{T_R - T_N}{T_R}\right)$ for $v = 0.01c$

a) 1.25×10^{-5} b) 5×10^{-5} c) 7.5×10^{-5} d) 1.0×10^{-4}

Sol^{no}

$$T_N = \frac{1}{2} m_0 v^2$$

$$T_R = mc^2 - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2$$

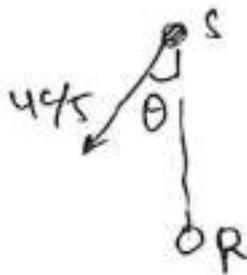
$$= \frac{m_0 c^2}{\sqrt{1 - (0.01)^2}} - m_0 c^2 = m_0 c^2 \left(\frac{1}{\sqrt{1 - 10^{-4}} - 1} \right)$$

$$\frac{T_R - T_N}{T_R} = 1 - \frac{T_N}{T_R} = 1 - \frac{\frac{1}{2} v^2 \cdot m_0}{m_0 c^2 \left(\frac{1}{\sqrt{1 - 10^{-4}} - 1} \right)}$$

$$\frac{T_R - T_N}{T_R} = \frac{1 - \frac{(0.01)^2}{2}}{\left(\frac{1}{\sqrt{1-10^{-4}}} - 1\right)} = 7.5 \times 10^{-5}$$

C

Ques A distinct source, emitting radiation of frequency ω , moves with a velocity $4c/5$ in a certain direction w.r.t a receiver.



The upper cut off frequency of the receiver is $3\omega/2$. Let θ be the angle as shown. For the receiver to detect the radiation,

θ should be at least:

- a) $\cos^{-1}\left(\frac{1}{2}\right)$ b) $\cos^{-1}\left(\frac{3}{4}\right)$ c) $\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ d) $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$

Sol^{no}

$$\omega' = \frac{\omega \sqrt{1 - v^2/c^2}}{1 - \frac{v}{c} \cos \theta}$$

$$\frac{3\omega}{2} = \frac{\omega \sqrt{1 - \left(\frac{4}{5}\right)^2}}{1 - \frac{4}{5} \cos \theta}$$

$$1 - \frac{4}{5} \cos \theta = \frac{2}{3} \times \frac{3}{5}$$

$$\cos \theta = \frac{3}{4} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{4}\right) \quad \text{B}$$

Q113 13 let (x, t) & (x', t') be the coordinate system (6)
 used by the observer O and O' , respectively. observer
 O' moves with a velocity $v = \beta c$ along their common
 positive x -axis. If $x_+ = x + ct$ and $x_- = x - ct$ are the
 linear combinations of the coordinates, the Lorentz
 transformation relating O & O' takes the form:

$$a) x'_+ = \frac{x_- - \beta x_+}{\sqrt{1 - \beta^2}} \quad \text{and} \quad x'_- = \frac{x_+ - \beta x_-}{\sqrt{1 - \beta^2}}$$

$$b) x'_+ = \sqrt{\frac{1 + \beta}{1 - \beta}} x_+ \quad \text{and} \quad x'_- = \sqrt{\frac{1 - \beta}{1 + \beta}} x_-$$

$$c) x'_+ = \frac{x_+ + \beta x_-}{\sqrt{1 - \beta^2}} \quad \text{and} \quad x'_- = \frac{x_- - \beta x_+}{\sqrt{1 - \beta^2}}$$

$$d) x'_+ = \sqrt{\frac{1 - \beta}{1 + \beta}} x_+ \quad \text{and} \quad x'_- = \sqrt{\frac{1 + \beta}{1 - \beta}} x_-$$

Solⁿg- $x'_+ = x' + ct'$

$$x'_+ = \gamma(x - vt) + c\left(t - \frac{vx}{c^2}\right)\gamma$$

$$= \gamma \left[x + ct - \frac{v}{c}(x + ct) \right]$$

$$= \frac{1}{\sqrt{1 - \beta^2}} [x_+ - \beta x_+] = \frac{1}{\sqrt{1 - \beta^2}} [1 - \beta] x_+$$

$$x'_+ = \sqrt{\frac{1 - \beta}{1 + \beta}} x_+ \quad \text{Similarly } x'_- = \sqrt{\frac{1 + \beta}{1 - \beta}} x_-$$

□

Quesy For a particle of energy E and momentum p (in frame f), the rapidity y is defined as (62)

$$y = \frac{1}{2} \ln \left(\frac{E + p_3 c}{E - p_3 c} \right)$$

In frame f' moving with velocity $v = (0, 0, \beta c)$ w.r.t f , the rapidity y' will be:

- a) $y' = y + \frac{1}{2} \ln(1 - \beta^2)$ b) $y' = y - \frac{1}{2} \ln \left(\frac{1 + \beta}{1 - \beta} \right)$
 c) $y' = y + \ln \left(\frac{1 + \beta}{1 - \beta} \right)$ d) $y' = y + 2 \ln \left(\frac{1 + \beta}{1 - \beta} \right)$

Sol^{no} - $y' = \frac{1}{2} \ln \left(\frac{E' + p'_3 c}{E' - p'_3 c} \right)$ in f'

$$y' = \frac{1}{2} \ln \left[\frac{E - p_3 v + p_3 c - \frac{E v}{c}}{E - p_3 v - p_3 c + \frac{E v}{c}} \right] \quad \begin{cases} \rightarrow E' = (E - v p) \gamma \\ \rightarrow p' = (p - \frac{v E}{c^2}) \gamma \end{cases}$$

$$y' = \frac{1}{2} \ln \left[\frac{(E + p_3 c) - \frac{v}{c} (E + p_3 c)}{(E - p_3 c) + \frac{v}{c} (E - p_3 c)} \right]$$

$$y' = \frac{1}{2} \ln \left[\left(\frac{E + p_3 c}{E - p_3 c} \right) \frac{(1 - \frac{v}{c})}{(1 + \frac{v}{c})} \right]$$

$$= \frac{1}{2} \ln \left(\frac{E + p_3 c}{E - p_3 c} \right) + \frac{1}{2} \ln \left(\frac{1 - \beta}{1 + \beta} \right)$$

$$y' = y + \frac{1}{2} \ln \left(\frac{1 - \beta}{1 + \beta} \right) \Rightarrow y' = y - \frac{1}{2} \ln \left(\frac{1 + \beta}{1 - \beta} \right) \quad \boxed{B}$$

Ques 15 A relativistic particle moves with a constant velocity v w.r.t the laboratory frame. In time τ , measured in the rest frame of the particle, the distance that it travels in the laboratory frame is:

- a) $v\tau$
- b) $\frac{c\tau}{\sqrt{1-v^2/c^2}}$
- c) $v\tau\sqrt{1-v^2/c^2}$
- d) $\frac{v\tau}{\sqrt{1-v^2/c^2}}$

Sol^{no}:-

$$t = \frac{\tau}{\sqrt{1-v^2/c^2}}$$

distance = speed \times time = $\frac{v\tau}{\sqrt{1-v^2/c^2}}$ [D]

Ques 16 An inertial observer sees two events E_1 & E_2 happening at the same location but 6 μ s apart in time. Another observer moving with a constant velocity v (w.r.t to one) sees the same event to be 9 μ s apart. The spatial distance b/w the events, as measured by and observed is:

- a) 300m
- b) 1000m
- c) 8000m
- d) 2700m

Sol^{no}:-

$$\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}} \Rightarrow 9 = \frac{6}{\sqrt{1-v^2/c^2}} \Rightarrow v = \frac{\sqrt{5}}{3}c$$

$$\Delta x' = \frac{\Delta x + v\Delta t'}{\sqrt{1-v^2/c^2}} = \frac{0 + \frac{\sqrt{5}}{3} \times 3 \times 10^8 \times 9 \times 10^{-6}}{\sqrt{1-\frac{5}{9}}} = 8000m$$

[C]

ERROR ANALYSIS

(64)

Deviation in the value of any variable from the standard value.

Absolute error:

$$\Delta x = (\text{measured} - \text{True}) \text{ value}$$

Δx has unit of x

Relative error:

$$\frac{\Delta x}{x} = \left(\frac{\text{measured value} - \text{True value}}{\text{True value}} \right) \times 100$$

unitless Parameter.

Percentage error:

$$\% \text{ error} = \frac{\Delta x}{x} \times 100 \%$$

maximum error (limiting error):

1) Errors in the variable are co-related:

Two or more variables are dependent on each other.

$$\text{let } z = x \pm y$$

$$\Delta z = \Delta x + \Delta y \rightarrow \text{absolute error}$$

$$\text{let } z = x \cdot y \text{ OR } x/y$$

$$\ln z = \ln x + \ln y$$

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y} \rightarrow \text{Relative error.}$$

*) Errors in the variable are uncorrelated:

Two or more variables are independent of each other.

$$\text{let } z = x \pm y$$

$$\Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

↑
uncertainty in error (Best approximation error)

$$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

↑
Root mean square error.

Ques! The acceleration due to gravity g is determined by measuring the time period T & the length L of the simple pendulum. If the uncertainties in the measurement of T & L are ΔT & ΔL respectively, the fractional error $\Delta g/g$ in measuring g is best approximated by:

a) $\frac{|\Delta L|}{L} + \frac{|\Delta T|}{T}$

b) $\frac{|\Delta L|}{L} + \frac{|2\Delta T|}{T}$

c) $\sqrt{\frac{|\Delta L|}{L} + \frac{|\Delta T|}{T}}$

d) $\sqrt{\left(\frac{\Delta L}{L}\right)^2 + \left(\frac{2\Delta T}{T}\right)^2}$

solⁿo- $T = 2\pi \sqrt{\frac{l}{g}} \Rightarrow g = 4\pi^2 \frac{l}{T^2}$

Best approximation errors

(66)

$$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

□

Ques 2 In a measurement of the viscous drag force experienced by spherical particles in a liquid, the force is found to be proportional to $v^{1/3}$ where v is the measured volume of each particle. If v is measured to be 30 mm^3 , with an uncertainty of 2.7 mm^3 , the resulting relative percentage uncertainty in the measured force is:

- a) 2.08 b) 0.09 c) 6 d) 3

Soln $F \propto v^{1/3}$ (given)

$$\ln F = \ln c + \frac{1}{3} \ln v$$

$$\frac{\Delta F}{F} = 0 + \frac{1}{3} \frac{\Delta v}{v}$$

$$\% \text{ error in } F: \frac{\Delta F}{F} \times 100\% = \frac{1}{3} \times \frac{2.7}{30} \times 100\%$$

$$= 3\% \quad \square$$

Ques 3 The viscosity η of a liquid is given by:

$$\eta = \frac{\pi P a^4}{8 l v} \quad (\text{Poiseuille's formula})$$

Assume that l & v can be measured very accurately,

but the pressure P has an rms error of 1%. & (67)
 radius a has an independent rms error of 3%,
 the rms error of the viscosity is closest to:

- a) 2%. b) 4%. c) 12%. d) 13%.

Solⁿg- rms error = $\sqrt{\left(\frac{\Delta P}{P}\right)^2 + \left(4 \frac{\Delta a}{a}\right)^2}$
 $= \sqrt{1 + 16(3)^2} = \sqrt{145}$
 $\approx 12\%.$ C

Ques 4 Two parallel plate capacitors, separated by distance x and $1.1x$, respectively have a dielectric material of dielectric constant 3.0 inserted b/w the plates and are connected to a battery of voltage V . The difference in charge on the second capacitor compared to the first is:

- a) 66%. b) 20%. c) -3.3%. d) -10%.

Solⁿg- $Q_1 = C_1 V_1 = \frac{3\epsilon_0 A}{x} V$

$$Q_2 = C_2 V_2 = \frac{3\epsilon_0 A}{1.1x} V$$

$$\left(\frac{\Delta Q}{Q_1} \times 100\right)\% = \left(\frac{Q_2 - Q_1}{Q_1}\right) \times 100\%.$$

$$= \frac{\frac{1}{1.1} - 1}{1} \times 100\% \approx -10\%.$$

D

Que 5 A resistance is measured by passing current through it and measuring the resulting voltage drop. If the voltmeter and the ammeter have uncertainty of 3% & 4%, respectively, then

A) The uncertainty in value of resistance is:

- a) 7.0% b) 3.5% c) 5.0% d) 12.0%

Solⁿo- $V = IR \Rightarrow R = \frac{V}{I}$

$$U_R = \sqrt{\left(\frac{\Delta V}{V} \times 100\right)^2 + \left(\frac{\Delta I}{I} \times 100\right)^2}$$

$$= \sqrt{3^2 + 4^2} = 5\% \quad \boxed{C}$$

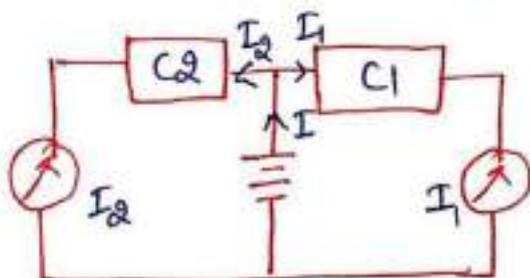
B) The uncertainty in computed value of the power dissipated in resistance is:

- a) 7% b) 5% c) 11% d) 9%

Solⁿo- Power: $P = VI$ or $I^2 R$

$$U_P = \sqrt{3^2 + 4^2} = 5\% \quad \boxed{B}$$

Que 6 A battery powers two circuits C_1 & C_2 as shown in the figure:



The total current I drawn from the battery is estimated

by measuring the currents I_1 & I_2 through the individual circuits. If I_1 & I_2 both are 200mA & if the errors in the measurement are 3mA & 4mA respectively, the error in the estimate of I is:

- a) 7.0 mA b) 5.0 mA c) 7.5 mA d) 10.5 mA

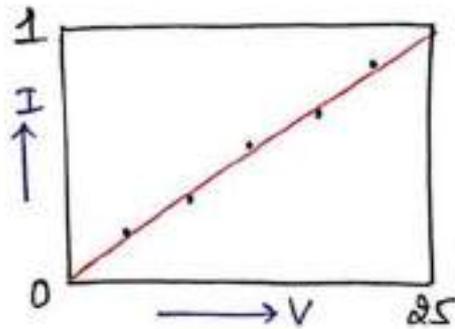
Sol^{no} $P_t = P_1 + P_2$
 $= I_1^2 R_1 + I_2^2 R_2$
 $I^2 = I_1^2 + I_2^2$
 $\Delta I = \sqrt{(\Delta I_1)^2 + (\Delta I_2)^2}$
 $= \sqrt{9 + 16} = 5 \text{ mA}$ **B**

Que 7 The experimentally measured values of the variables x and y are 2.00 ± 0.05 & 3.00 ± 0.02 respectively. what is the error in the calculated value of $Z = 3y - 2x$ from the measurement?

- a) 0.12 b) 0.05 c) 0.03 d) 0.07

Sol^{no} $Z = 3y - 2x$
 $\Delta Z = \sqrt{\left(3 \frac{\partial Z}{\partial y} \Delta y\right)^2 + \left(-2x \frac{\partial Z}{\partial x} \Delta x\right)^2}$
 $= \sqrt{(3 \times 0.02)^2 + (-2 \times 0.05)^2}$
 $= 0.12$ **A**

Ques Both the data points and a linear fit to the current vs voltage of a resistor are shown in the graph below. (10)



If the error in the slope is $1.225 \times 10^{-3} \Omega^{-1}$, then the value of resistance estimated from the graph is:

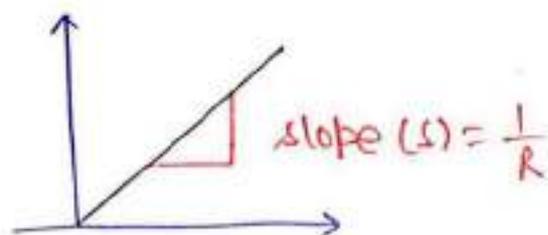
a) $(0.04 \pm 0.8) \Omega$

b) $(25 \pm 0.8) \Omega$

c) $(25 \pm 1.25) \Omega$

d) $(25 \pm 0.0125) \Omega$

Solⁿ:- $R = \frac{V}{I} = \frac{25}{1} = 25 \Omega$



$$\frac{\Delta R}{R} = \frac{\Delta s}{s} \Rightarrow \Delta R = \frac{\Delta s}{s} \times R$$

$$\Delta R = 1.225 \times 10^{-3} \times \frac{1}{1/25} \times 25$$

$$\Delta R = 625 \times 1.225 \times 10^{-3} = 0.8 \Omega$$

$$R = (25 \pm 0.8) \Omega$$

B

Que⁹ 1 gm of salt is dissolved in water that is filled to a height of 5 cm in a beaker of diameter 10 cm. The accuracy of length measurement is 0.01 cm while that of mass measurement is 0.01 gm. When measuring the concentration c , the fractional error $\Delta c/c$ is: (71)

a) 0.8% b) 0.14% c) 0.5% d) 0.28%

Solⁿg-
$$\begin{array}{l|l|l} m = 1 \text{ gm} & R = 5 \text{ cm} & d = 10 \text{ cm} \\ \Delta m = 0.01 \text{ gm} & \Delta R = 0.01 \text{ cm} & \Delta d = 0.01 \text{ cm} \end{array}$$

$$\text{concentration } (c) = \frac{\text{mass}}{\text{volume}} = \frac{m}{V}$$

$$\text{volume} = \pi r^2 h = \pi \left(\frac{d}{2}\right)^2 R$$

$$\begin{aligned} \frac{\Delta V}{V} &= \sqrt{\left(2 \frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta R}{R}\right)^2} \\ &= \sqrt{4 \times 10^{-6} + \frac{(0.01)^2}{25}} = 2\sqrt{2} \times 10^{-3} \end{aligned}$$

$$\frac{\Delta m}{m} = \frac{0.01 \times 10^{-3}}{1} = 10^{-5}$$

$$\begin{aligned} \left(\frac{\Delta c}{c} \times 100\%\right) &= \sqrt{\left(\frac{\Delta m}{m} \times 100\right)^2 + \left(\frac{\Delta V}{V} \times 100\right)^2} \\ &= 0.28\% \quad \boxed{D} \end{aligned}$$

Ques 10 The decay constants f_p of the heavy pseudoscalar mesons, in the heavy quark limit, are related to their masses m_p by the relation $f_p = a/\sqrt{m_p}$ where a is an empirical parameter to be determined. The values:

$$m_p = 6400 \pm 160 \text{ MeV}$$

$$f_p = 180 \pm 15 \text{ MeV}$$

corresponds to uncorrelated measurements of a meson. The error on the estimated of a is:

a) $175 \text{ (MeV)}^{3/2}$

b) $900 \text{ (MeV)}^{3/2}$

c) $1200 \text{ (MeV)}^{3/2}$

d) $2400 \text{ (MeV)}^{3/2}$

Solⁿ:- $f_p = \frac{a}{\sqrt{m_p}} \Rightarrow a = f_p \times \sqrt{m_p}$

$$\text{error} = 15 \text{ MeV} \times \frac{1}{2} \times (160) \text{ (MeV)}^{1/2}$$

$$= 1200 \text{ (MeV)}^{3/2}$$



Ques 11 The first order diffraction peak of a crystalline solid occurs at a scattering angle of 30° when the diffraction pattern is recorded using an X-ray beam of wavelength 0.15 nm . If the error in the measurements of wavelength and angle are 0.01 nm and 1° respectively, then the error in calculating error in interplaner spacing will be approx be:

a) $1.1 \times 10^{-2} \text{ mm}$

b) $1.3 \times 10^{-4} \text{ mm}$

c) $2.5 \times 10^{-2} \text{ nm}$

d) $2.0 \times 10^{-3} \text{ nm}$

soln -

$$d \sin \theta = m \lambda$$

(73)

$$d = \frac{m \lambda}{\sin \theta} = \frac{m \lambda}{x}$$

$$x = \sin \theta$$

$$\Delta x = \cos \theta \Delta \theta$$

$$\frac{\Delta d}{d} = \sqrt{\left(\frac{\Delta \lambda}{\lambda}\right)^2 + \left(\frac{\cos \theta}{\sin \theta} \Delta \theta\right)^2}$$

$$= \sqrt{\left(\frac{0.01}{0.15}\right)^2 + (\cot 30^\circ)^2 \left(\frac{1^\circ \pi}{180}\right)^2}$$

$$= \sqrt{\frac{1}{225} + 3 \left(\frac{3.14}{180}\right)^2}$$

$$\frac{\Delta d}{d} = 7.32 \times 10^{-2}$$

$$\Delta d = (7.32 \times 10^{-2}) \times 0.15 \text{ nm}$$
$$= 1.1 \times 10^{-2} \text{ nm.}$$

A